

INTEGRAL CANONICAL MODELS FOR SPIN SHIMURA VARIETIES

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ABSTRACT. We construct regular integral canonical models for Shimura varieties attached to Spin groups at (possibly ramified) primes $p > 2$ where the level is not divisible by p . We exhibit these models as schemes of 'relative PEL type' over integral canonical models of larger Spin Shimura varieties with good reduction at p . Work of Vasiu-Zink then shows that the classical Kuga-Satake construction extends over the integral model and that the integral models we construct are canonical in a very precise sense. We also construct good compactifications for our integral models. Our results have applications to the Tate conjecture for K3 surfaces, as well as to Kudla's program of relating intersection numbers of special cycles on orthogonal Shimura varieties to Fourier coefficients of modular forms.

INTRODUCTION

The object of study in this paper is a Shimura variety $\mathrm{Sh}_K(G, X)$, where $G = \mathrm{GSpin}(V, Q)$ is attached to a quadratic space (V, Q) over \mathbb{Q} of signature $(n, 2)$. More precisely, suppose that $L \subset V$ is a maximal lattice with dual lattice L^\vee . We work over $\mathbb{Z}[\frac{1}{2}]$ and with level $K \subset G(\mathbb{A}_f)$, where K is the largest sub-group of the stabilizer of $L_{\widehat{\mathbb{Z}}}$ that acts trivially on L^\vee/L . Our results can be summarized by the following theorem.

Theorem 1. *Over $\mathbb{Z}[\frac{1}{2}]$, $\mathrm{Sh}_K(G, X)$ admits a regular canonical model with a regular toroidal compactification.*

The word 'canonical' is used in a very precise sense that is explained in § 6.

The basic idea of the proof is quite simple. Let us work over $\mathbb{Z}_{(p)}$ for some odd prime p . When the lattice L is self-dual at a prime p , K_p is hyperspecial, and the results of [Kis10] already give us a smooth integral canonical model over $\mathbb{Z}_{(p)}$ (compactifications are dealt with in [MP13a]). In general, we can exhibit L as a sub-lattice of a bigger lattice \tilde{L} that is self-dual at p , and is such that the associated quadratic space (\tilde{V}, \tilde{Q}) over \mathbb{Q} again has signature $(m, 2)$, for some $m \in \mathbb{Z}_{\geq 0}$. This in turn allows us to exhibit $\mathrm{Sh}_K(G, X)$ as an intersection of divisors in $\mathrm{Sh}_{\tilde{K}}(\tilde{G}, \tilde{X})$, where (\tilde{G}, \tilde{X}) is the Spin Shimura datum attached to (\tilde{V}, \tilde{Q}) . Let $\tilde{\mathcal{S}}_{\tilde{K}}$ be the integral canonical model for $\mathrm{Sh}_{\tilde{K}}(\tilde{G}, \tilde{X})$ over $\mathbb{Z}_{(p)}$. We show that the divisors have a moduli interpretation, which allows us to define models for them over $\tilde{\mathcal{S}}_{\tilde{K}}$. We then construct our regular models as intersections (over $\tilde{\mathcal{S}}_{\tilde{K}}$) of these integral models of the divisors.

Let us be a little more precise. The classical Kuga-Satake construction, combined with Kisin's construction, gives us a natural polarized abelian scheme $(\tilde{A}^{\mathrm{KS}}, \lambda^{\mathrm{KS}})$ over $\tilde{\mathcal{S}}_{\tilde{K}}$. Let $\tilde{\mathbf{L}}_B$ be the $\mathbb{Z}_{(p)}$ -local system over $\tilde{\mathcal{S}}_{\tilde{K}, \mathbb{C}}$ attached to $\tilde{L}_{\mathbb{Z}_{(p)}}$. Let $\tilde{\mathbf{H}}_B$ be the first Betti cohomology of $\tilde{A}_{\mathbb{C}}^{\mathrm{KS}}$ over $\tilde{\mathcal{S}}_{\tilde{K}, \mathbb{C}}$ with coefficients in $\mathbb{Z}_{(p)}$. Then the construction allows us to view $\tilde{\mathbf{L}}_B$ as a sub-local system of $\tilde{\mathbf{H}}_B \otimes \tilde{\mathbf{H}}_B^\vee$. Given an $\mathrm{Sh}_{\tilde{K}}(\tilde{G}, \tilde{X})$ -scheme T , we say that an endomorphism f of $\tilde{A}_T^{\mathrm{KS}}$ is **special** if over $T_{\mathbb{C}}$ it induces a section of $\tilde{\mathbf{L}}_B$. Then the divisors mentioned above can be viewed as the loci where \tilde{A}^{KS} inherits certain special endomorphisms of fixed degree. So we obtain a moduli interpretation for $\mathrm{Sh}_K(G, X) \rightarrow \mathrm{Sh}_{\tilde{K}}(\tilde{G}, \tilde{X})$ as the locus over which \tilde{A}^{KS} inherits a certain family of special endomorphisms. For an analytic viewpoint of all this in the case of orthogonal Shimura varieties, cf. [Kud97].

Let $\widetilde{\mathbf{H}}_{\text{cris}}$ be the first crystalline cohomology crystal of $\widetilde{A}_{\mathbb{F}_p}^{\text{KS}}$ over $\widetilde{\mathcal{S}}_{\widetilde{K}, \mathbb{F}_p}$. Then Kisin's work provides us with a canonical sub-crystal $\widetilde{\mathbf{L}}_{\text{cris}} \subset \widetilde{\mathbf{H}}_{\text{cris}} \otimes \widetilde{\mathbf{H}}_{\text{cris}}^{\vee}$ attached to the quadratic space L . This allows us to give a definition of specialness in characteristic p as well: For any $\widetilde{\mathcal{S}}_{\widetilde{K}, \mathbb{F}_p}$ -scheme T , an endomorphism f of $\widetilde{A}_T^{\text{KS}}$ is **special** if its crystalline realization is a section of $\widetilde{\mathbf{L}}_{\text{cris}}$ at every point of T . We can patch together the two notions of specialness to get the notion of a special endomorphism of $\widetilde{A}^{\text{KS}}$ in general. This permits us to extend the moduli interpretation for $\text{Sh}_K(G, X)$ (relative to $\text{Sh}_{\widetilde{K}}(\widetilde{G}, \widetilde{X})$) over $\mathbb{Z}_{(p)}$, and so gives us a natural integral model for $\text{Sh}_K(G, X)$. To study its local properties, it remains to study the problem of deforming special endomorphisms of A^{KS} . This we do using ideas of Deligne [Del81] and Ogus [Ogu79], though we have preferred to couch it in the language of local models.

Note that, for low values of n , there are direct, moduli-theoretic ways to construct these integral canonical models. For the case of Shimura curves, cf. [KRY06], and for the case of certain Hilbert-Blumenthal surfaces, cf. [KR99]. Moreover, there is work in progress by Kisin and Pappas, which will generalize the methods of [Kis10] to construct integral models of Shimura varieties of abelian type with general parahoric level at p . Though part of our work is, in some sense, subsumed by theirs, the simple and direct nature of our construction seems to be quite useful in applications, which include the Tate conjecture for K3 surfaces in odd characteristic [MP13b], and also forthcoming work, in collaboration with Andreatta, Goren and Howard, on the arithmetic intersection theory of spin Shimura varieties.

Tour of contents. In § 1, we summarize what we need about Clifford algebras and Spin groups. In § 2, after a review of quadratic lattices, we define the spaces that will serve as local models for our regular models, and study their properties.

In § 3, we discuss the spin Shimura variety and its integral canonical model at a prime where the level is hyperspecial. In particular, we describe carefully how the existence of the crystal $\widetilde{\mathbf{L}}_{\text{cris}}$ discussed above follows from Kisin's results in [Kis10].

§ 4 is a discussion of special endomorphisms in characteristic 0. In § 5, we define special endomorphisms in characteristic p and study their deformation theory.

Given this prelude, the construction of regular models in § 6 is quite simple. In fact—since we will need this generality in [MP13b]—we construct models for spin Shimura varieties attached to arbitrary quadratic lattices over $\mathbb{Z}_{(p)}$. When the lattice is maximal, we use results of Vasiu-Zink [VZ10] to show that these models are regular and canonical in a very precise sense.

In § 7, we prove an ℓ -independence result for special endomorphisms. More precisely, we show, under certain conditions, that the specialness of an endomorphism can be checked using its ℓ -adic realization, even for primes $\ell \neq p$. These conditions are now always known to hold by the work of Kisin.

Finally, in § 8, we show that the results of [MP13a] give us good toroidal compactifications of the regular canonical models, and, more generally, for the integral models attached to the quadratic lattices considered in § 6.

Acknowledgments. We thank Fabrizio Andreatta, Bhargav Bhatt, Ben Howard, Mark Kisin, George Pappas and Junecue Suh for helpful comments, insights and corrections. This work was partially supported by NSF Postdoctoral Research Fellowship DMS-1204165.

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1. CLIFFORD ALGEBRAS AND SPIN GROUPS

1.1. Let R be a commutative ring in which 2 is invertible, and let (L, Q) be a quadratic space over R : by this we mean a free R -module L of finite rank equipped with a quadratic form $Q : L \rightarrow R$. We will denote by $[\cdot, \cdot]_Q : L \otimes L \rightarrow R$ the associated symmetric bi-linear form, its relation with Q being given by the formula

$$[v, w]_Q = Q(v + w) - Q(v) - Q(w).$$

We have the associated **Clifford algebra** $C := C(L)$ over R , equipped with an embedding $L \hookrightarrow C(L)$. This algebra has the following universal property: For any R -algebra S we have a bijection:

$$\begin{aligned} \mathrm{Hom}_{R\text{-alg}}(C, S) &\xrightarrow{\sim} \{\varphi \in \mathrm{Hom}_R(L, S) : \varphi(v)^2 = Q(v)\} \\ f &\mapsto f|_L. \end{aligned}$$

From this universal property, it is clear that, if S is an R -algebra, and (L_S, Q_S) is the quadratic space over S obtained from (L, Q) by extension of scalars, then we have a canonical isomorphism of S -algebras

$$C_S := C \otimes_R S \xrightarrow{\sim} C(L_S).$$

We denote by $j : C \rightarrow C$ the involution corresponding to the map

$$\begin{aligned} L &\rightarrow C \\ v &\mapsto -v; \end{aligned}$$

and we denote by $*$: $C \rightarrow C^{\mathrm{op}}$ the anti-involution, called the **canonical involution**, corresponding to the map

$$\begin{aligned} L &\rightarrow C^{\mathrm{op}} \\ v &\mapsto v. \end{aligned}$$

We have a $\mathbb{Z}/2\mathbb{Z}$ -grading

$$C = C^+ \oplus C^-$$

where C^+ is the $+1$ -eigenspace of j and C^- is the -1 eigenspace of j . C^+ is a sub-algebra of C , and we will call it the **even Clifford algebra**.

Lemma 1.2. *Suppose that $L_1 \subset L$ is a direct summand of co-dimension r . Then C is a projective module of rank 2^r over $C(L_1)$.*

Proof. Choose any decomposition of R -modules $L = L_1 \oplus L_2$; then one easily checks that the natural multiplication map

$$C(L_1) \otimes_R C(L_2) \rightarrow C$$

is an isomorphism of $C(L_1)$ -modules. Now, we only have to note that $C(L_2)$ is projective over R of rank 2^r : Indeed, it is isomorphic as an R -module to the exterior algebra $\wedge^\bullet L_2$. \square

1.3. Suppose now that Q is non-degenerate: that is, it induces an isomorphism $L \xrightarrow{\sim} L^\vee$. Then we can use the Clifford algebra to define a reductive group scheme $\mathrm{GSpin}(L, Q)$ over R . For any R -algebra S , we have:

$$\mathrm{GSpin}(L, Q)(S) = \{x \in (C_S^+)^{\times} : x(L_S)x^{-1} = L_S\}.$$

It sits in a cross with exact lines:

$$\begin{array}{ccccc} & & \mathrm{Spin}(L, Q) & & \\ & & \downarrow & & \\ \mathbb{G}_m & \hookrightarrow & \mathrm{GSpin}(L, Q) & \xrightarrow{x \mapsto (v \mapsto xvx^{-1})} & \mathrm{SO}(L, Q) \\ & & \downarrow \nu : x \mapsto x^*x & & \\ & & \mathbb{G}_m & & \end{array}$$

$\mathrm{GSpin}(L, Q)$ acts naturally on C via left translation. When we view C as a representation of $\mathrm{GSpin}(L, Q)$ under left multiplication, we will denote it by H . Denote by $\rho : C \rightarrow \mathrm{End}_R(H)^{\mathrm{op}}$ the right action of C on H by right multiplication. The canonical embedding $L \subset C$ allows us to view L as a space of ρ -equivariant endomorphisms of H .

From now on, we will continue to maintain the assumption that Q is non-degenerate.

Remark 1.4. In the literature, one sometimes also finds the additional condition $x^*x \in S^\times$ as part of the definition of $\mathrm{GSpin}(L, Q)(S)$. This condition is in fact redundant. This is well known when S is a field (cf. for example, [Shi10, Theorem 24.7]), and from this the redundancy follows by standard arguments: If S is reduced, then the condition can be checked after specializing at all points of S , and so holds over S itself. In general, after a faithfully flat base change, we can assume that $(L, Q) = (L_0, Q_0) \otimes_{\mathbb{Z}} R$, for some self-dual quadratic space (L_0, Q_0) over $\mathbb{Z}[\frac{1}{2}]$. Now, one only needs to check the condition $x^*x \in S^\times$, when S is the ring of functions of $\mathrm{GSpin}(L_0, Q_0)$. It is therefore enough to observe that S is reduced: Indeed, it is a finitely generated $\mathbb{Z}[\frac{1}{2}]$ -algebra all of whose fibers are smooth of the same dimension, and must therefore itself be smooth.

1.5. Let $\mathrm{GL}_R^+(H) \subset \mathrm{GL}_R(H)$ be the sub-group¹ of automorphisms that preserve the grading on H : this is the group of units in the algebra $\mathrm{End}_R^+(H)$ of graded endomorphisms of H . The commutator of $\rho(C)$ in $\mathrm{GL}_R^+(H)$ is the group scheme $\mathrm{U}(H)$, where, for any R -algebra S , we have $\mathrm{U}(H)(S) = (\mathrm{End}_{\rho(C)}^+(H_S))^{\times} = (C_S^+)^{\times}$. By definition, $\mathrm{GSpin}(L, Q)$ is a sub-group of $\mathrm{U}(H)$.

The pairing $[\varphi_1, \varphi_2] = \frac{1}{2^n} \mathrm{Tr}(\varphi_1 \circ \varphi_2)$ on $\mathrm{End}_R(H)$ is symmetric, non-degenerate and restricts to the pairing $[\cdot, \cdot]_Q$ on $L \subset C$. Choose an R -basis e_1, \dots, e_m of L and let $(r_{i,j}) \in \mathrm{GL}_m(R)$ be the matrix whose inverse is $(r_{i,j})^{-1} = ([e_i, e_j]_Q)$. Consider the endomorphism $\pi : \mathrm{End}_R(H) \rightarrow \mathrm{End}_R(H)$ given, for $\varphi \in \mathrm{End}_R(H)$ by

$$\pi(\varphi) = \sum_{i,j} r_{i,j} \cdot [\varphi, e_i] \cdot e_j.$$

It is a simple check to see that π does not depend on the choice of basis for L . The following lemma is also easily shown.

Lemma 1.6.

(1) π is a $\mathrm{GSpin}(L, Q)$ -equivariant idempotent projector and its image is $L \subset \mathrm{End}_R(H)$.

¹The word ‘group’ from now on will mean ‘ R -group scheme’

(2) $\mathrm{GSpin}(L, Q) \subset \mathrm{U}(H)$ is the stabilizer of $\pi : \mathrm{End}_R(H) \rightarrow \mathrm{End}_R(H)$.

□

Remark 1.7. In the above situation, suppose that $R = \mathbb{Z}_{(p)}$ and only that $(L_{\mathbb{Q}}, Q)$ is non-degenerate. Then it is easy to see that, even if π no longer preserves $\mathrm{End}_{\mathbb{Z}_{(p)}}(H) \subset \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$, the image of $\mathrm{End}_{\mathbb{Z}_{(p)}}(H)$ under π is exactly the dual lattice $L^{\vee} \subset L_{\mathbb{Q}}$. We will return to this in § 3.

1.8. There exists an R -linear **reduced trace map** $\mathrm{Trd} : C \rightarrow R$, unique up to scaling by R^{\times} , such that the pairing

$$(x, y) \mapsto \mathrm{Trd}(xy)$$

is a non-degenerate symmetric bilinear form on C . This is easily checked when (L, Q) admits a Lagrangian decomposition $L = W \oplus W'$, via the isomorphisms

$$C \simeq \begin{cases} \mathrm{End}_R(\bigwedge^{\bullet} H), & \text{if } \mathrm{rk}_R L \text{ is even;} \\ \mathrm{End}_R(\bigwedge^{\bullet} W) \times \mathrm{End}_R(\bigwedge^{\bullet} W'), & \text{if } \mathrm{rk}_R L \text{ is odd.} \end{cases}$$

Since, fppf locally on $\mathrm{Spec} R$, we can always find a Lagrangian decomposition of L , we can conclude in general by fppf descent.

For any $\delta \in C^{\times}$ such that $\delta^* = -\delta$, the form $\psi_{\delta}(x, y) = \mathrm{Trd}(x\delta y^*)$ defines an R -valued symplectic form on H .

Lemma 1.9. *For δ as above, the line $[\psi_{\delta}]$ in $\mathrm{Hom}(H \otimes H, R)$, spanned by the symplectic form ψ_{δ} , is preserved by $\mathrm{GSpin}(L, Q)$. The similitude character of $\mathrm{GSpin}(L, Q)$ obtained from its action on this line agrees with the spinor norm.*

□

Definition 1.10. The embedding $\mathrm{GSpin}(L, Q) \hookrightarrow \mathrm{GSp}(H, \psi_{\delta})$ (for any choice of δ as above) will be called a **Kuga-Satake embedding**.

Remark 1.11. We can also consider the global setting of a $\mathbb{Z}[\frac{1}{2}]$ -scheme S and a vector bundle L over S equipped with a quadratic form. All considerations and definitions above generalize immediately to this situation.

Definition 1.12. We will need one further piece of notation: For any pair of positive integers (r, s) , we set

$$H^{\otimes(r,s)} = \underbrace{H^{\vee} \otimes \cdots \otimes H^{\vee}}_{r \text{ times}} \otimes \underbrace{H \otimes \cdots \otimes H}_{s \text{ times}}$$

We will also use this notation for objects in other tensor categories without comment.

Note that we can now think of π as an element of $H^{\otimes(2,2)}$.

1.13. Let $G_0 = \mathrm{SO}(L, Q)$. Since there exists a central isogeny $G \rightarrow G_0$ of reductive groups, there is a bijective correspondence between parabolic sub-groups of G and those of G_0 . We want to make this correspondence explicit on the level of linear algebra, for certain parabolic sub-groups. To each isotropic sub-space $L^1 \subset L$, we can attach the parabolic sub-group $P_0 \subset G_0$ that stabilizes L^1 . We get a decreasing filtration:

$$0 = F^2 L \subset F^1 L = L^1 \subset F^0 L = (L^1)^{\perp} \subset F^{-1} L = L.$$

Since L^1 is isotropic, we have a canonical embedding of R -algebras

$$\bigwedge^{\bullet} L^1 \hookrightarrow C \hookrightarrow \mathrm{End}_R(H).$$

If $N \subset \text{End}_R(H)$, write $\text{im } N$ for the union of the images in H of the endomorphisms in N . Similarly, write $\text{ker } N$ for the intersection of the kernels of the elements of N . Then we have, for every integer $i = 0, \dots, r+1$:

$$\text{im}(\wedge^i L^1) = \text{ker}(\wedge^{r-i+1} L^1).$$

Moreover, $\text{im}(\wedge^i L^1) \subset \text{im}(\wedge^{i-1} L^1)$. So we can define a descending filtration $F^\bullet H$ on H by

$$F^i H = \text{im}(\wedge^i L^1).$$

Suppose that $\mu_0 : \mathbb{G}_m \rightarrow G_0$ is a co-character splitting $F^\bullet L$. It gives rise to a splitting

$$L = L^1 \oplus L^0 \oplus L^{-1},$$

with $F^i L = \bigoplus_{j \geq i} L^j$, and where $\mu_0(z)$ acts on L^i via z^i . In particular, L^{-1} is another isotropic direct summand of L that pairs non-degenerately with L^1 .

Take the increasing filtration $E_i H = \text{ker}(\wedge^{i+1} L^{-1}) = \text{im}(\wedge^{r-i} L^{-1})$, and set $H^i = E_i H \cap F^i H$. One easily checks that this is a splitting of $F^\bullet H$.

Let $\mu : \mathbb{G}_m \rightarrow \text{GL}(H)$ be the co-character that acts via z^i on H^i . By construction, H^i is $\rho(C)$ -stable, and one can check easily that $\mu(\mathbb{G}_m)$ preserves the grading on H . So μ must factor through $U(H)$. Furthermore, we find that, if $v \in L$ and $i = 0, 1, \dots, r$, then:

$$v \cdot H^i \subset \begin{cases} H^{i+1} & \text{if } v \in L^1; \\ H^i & \text{if } v \in L^0; \\ H^{i-1} & \text{if } v \in L^{-1}. \end{cases}$$

This shows that:

$$\mu(z)v\mu(z)^{-1} = \begin{cases} z \cdot v & \text{if } v \in L^1; \\ v & \text{if } v \in L^0; \\ z^{-1} \cdot v & \text{if } v \in L^{-1}. \end{cases}$$

In other words, μ factors through G and is a lift of μ_0 . This shows the following:

Proposition 1.14. *The parabolic sub-group $P \subset G$ lifting $P_0 \subset G_0$ is the stabilizer in G of $F^\bullet H$.*

□

2. LATTICES IN QUADRATIC SPACES

Our main reference for this section is [Shi10].

2.1. For any discrete valuation ring \mathcal{O} with fraction field F , and any quadratic space (Z, q) over F , an \mathcal{O} -lattice $M \subset Z$ is **maximal** if q is \mathcal{O} -valued on M , and there is no larger \mathcal{O} -lattice in Z on which q is \mathcal{O} -valued. For any \mathcal{O} -lattice $M \subset Z$ on which q is \mathcal{O} -valued, let $\text{disc}(M) = M^\vee/M$, where $M^\vee \subset Z$ is the dual lattice:

$$M^\vee = \{z \in Z : [m, z] \in \mathcal{O}, \text{ for all } m \in M\}.$$

We say that M is **self-dual** if $M = M^\vee$; any self-dual lattice is automatically maximal (since $p > 2$). Any quadratic space M over \mathcal{O} will be termed maximal (resp. self-dual or **perfect**) if it is maximal (resp. self-dual) in $M \left[\frac{1}{p} \right]$.

Let (L, Q) be a quadratic space over $\mathbb{Z}_{(p)}$ that is non-degenerate over \mathbb{Q} . We will assume that we are given a perfect quadratic space (\tilde{L}, \tilde{Q}) over $\mathbb{Z}_{(p)}$ admitting (L, Q) as a direct summand. Set $\Lambda = L^\perp \subset \tilde{L}$. Let $\tilde{G} = \text{GSpin}(\tilde{L}, \tilde{Q})$, and let $G \subset \tilde{G}$ be the sub-group such that, for any $\mathbb{Z}_{(p)}$ -algebra R , we have:

$$G(R) = \{g \in \tilde{G}(R) : g|_{\Lambda_R} \equiv 1\}.$$

Proposition 2.2.

- (1) The group scheme G is smooth over $\mathbb{Z}_{(p)}$ with generic fiber $\mathrm{GSpin}(L_{\mathbb{Q}}, Q)$.
- (2) Let R be any flat $\mathbb{Z}_{(p)}$ -algebra. Then

$$G(R) = \left\{ g \in G \left(R \left[\frac{1}{p} \right] \right) : gL_R = L_R \text{ and } g \text{ acts trivially on } L_R^{\vee}/L_R \right\}.$$

- (3) The group scheme G does not depend on the choice of perfect quadratic space \tilde{L} containing L .
- (4) Suppose that v_1, \dots, v_d is a basis for Λ . Let R be a $\mathbb{Z}_{(p)}$ -algebra, and suppose that we have a d -tuple $(w_1, \dots, w_d) \in \tilde{L}_R^d$, generating a direct summand of \tilde{L}_R and satisfying $[w_i, w_j]_Q = [v_i, v_j]_Q$, for all i, j . Then the functor $\mathcal{P}_{\underline{w}}$ on R -algebras given by

$$\mathcal{P}_{\underline{w}} : B \mapsto \left\{ g \in \tilde{G}(B) : gv_i = w_i, \text{ for all } i \right\}$$

is represented by a G_R -torsor over R .

Proof. For (1), note that $\mathbb{G}_m = \ker(\tilde{G} \rightarrow \mathrm{SO}(\tilde{L}, \tilde{Q}))$ is a smooth sub-group of G . Let G' (resp. \tilde{G}') be the $\mathbb{Z}_{(p)}$ -group scheme G/\mathbb{G}_m (resp. \tilde{G}/\mathbb{G}_m): it is enough to show that G' is smooth over $\mathbb{Z}_{(p)}$.

Let R be a $\mathbb{Z}_{(p)}$ -algebra, let $I \subset R$ be a square-zero ideal, and let $R_0 = R/I$. Suppose that we have $g_0 \in G'(R_0)$: We want to find $g \in G'(R)$ mapping to g_0 . Choose any $\tilde{g} \in \tilde{G}'(R)$ lifting g_0 . For $v \in \Lambda_R$, $\tilde{g}(v) - v$ belongs to $I \otimes_R \tilde{L}_R$. Consider the assignment $v \mapsto \tilde{g}(v) - v$: restricted to Λ_R , it factors through a map of R_0 -modules:

$$U : \Lambda_{R_0} \rightarrow I \otimes_{\mathbb{Z}_{(p)}} \tilde{L}.$$

It follows from its definition that U is an element of the R_0 -module $I \otimes_{\mathbb{Z}_{(p)}} \mathfrak{a}_{\Lambda}$, where

$$\mathfrak{a}_{\Lambda} = \{f \in \mathrm{Hom}(\Lambda, \tilde{L}) : [f(v), w] + [v, f(w)] = 0, \text{ for all } v, w \in \Lambda\}.$$

There is a natural restriction map $\mathrm{Lie}(\tilde{G}') \rightarrow \mathfrak{a}_{\Lambda}$. We claim that this is surjective. It is enough to check this after tensoring with \mathbb{F}_p , and here, one can do an easy dimension count. To finish the proof of (1), choose $X \in I \otimes_{\mathbb{Z}_{(p)}} \mathrm{Lie}(\tilde{G}')$ such that $X|_{\Lambda_{R_0}} = U$, and take $g = (1 - X)\tilde{g}$.

Let us show (2). Since R is flat over $\mathbb{Z}_{(p)}$, $G(R)$ is naturally contained in $G\left(R\left[\frac{1}{p}\right]\right)$. We just have to check that the functorial description of $G(R)$ identifies it with the subset described in (2).

This amounts to checking the following thing: Suppose that we have $g \in G\left(R\left[\frac{1}{p}\right]\right)$ such that $gL_R = L_R$. Then:

$$g\tilde{L}_R = \tilde{L}_R \Leftrightarrow g \text{ acts trivially on } L_R^{\vee}/L_R.$$

By hypothesis, g acts trivially on Λ_R . We also have a natural isomorphism:

$$\tilde{L}/\Lambda \xrightarrow{\sim} L^{\vee}.$$

Therefore, it is enough to show:

$$g\tilde{L}_R = \tilde{L}_R \Leftrightarrow (g - 1)(\tilde{L}_R) \subset L_R.$$

If $(g - 1)(\tilde{L}_R) \subset L_R$, then easily $g\tilde{L}_R = \tilde{L}_R$. So suppose that $g\tilde{L}_R \neq \tilde{L}_R$. Choose any $u \in \tilde{L}_R$; it is enough to show that $p^r(gu - u) \in L_R$, for some $r \in \mathbb{Z}_{\geq 0}$. Since $\Lambda \oplus L$ has finite

index in \tilde{L} , we can find $r \in \mathbb{Z}_{\geq 0}$ such that $p^r u = w + v$, for some $w \in \Lambda_R$ and $v \in L_R$, giving us the equality $p^r g u = g(w + v) = w + g v$, and so

$$p^r(gu - u) = w + gv - (w + v) = gv - v \in L_R.$$

This shows (2).

Take $R = \mathbb{Z}_{(p)}^{nr}$ to be the strict henselization of $\mathbb{Z}_{(p)}$. Then (3) is immediate from the following assertion: Given a smooth \mathbb{Q} -scheme X , a smooth $\mathbb{Z}_{(p)}$ -model \mathfrak{X} for X is uniquely determined by its set R -valued points $\mathfrak{X}(R) \subset X\left(R\left[\frac{1}{p}\right]\right)$. Indeed, this follows from [BT87, 1.7.6] and faithfully flat descent.

The proof of (4) proceeds similarly. We first show that $\mathcal{P}_{\underline{w}}$ is formally smooth: Suppose that $B \twoheadrightarrow B_0$ is a surjection of R -algebras with square-zero kernel I . Suppose also that we have an element $g_0 \in \mathcal{P}_{\underline{w}}(B_0)$. We can lift g_0 to an element $\tilde{g} \in \tilde{G}(B)$. For each i , $\tilde{g}v_i = w_i + u_i$, for some $u_i \in I \otimes \tilde{L}$. Just as above, we can find $X \in I \otimes \text{Lie } \tilde{G}$ such that $X(v_i) = \tilde{g}^{-1}u_i$. The element $g = \tilde{g}(1 - X)$ is now an element of $\mathcal{P}_{\underline{w}}$ lifting g_0 .

We can now finish by showing that, for every residue field k of R , $\mathcal{P}_{\underline{w}}(k)$ is non-empty. But this follows from Witt's extension theorem. Note that this is where we crucially use the hypothesis that the sub-space generated by $\{w_i\}$ is a direct summand. \square

Lemma 2.3. *Let (L, Q) be a maximal lattice over $\mathbb{Z}_{(p)}$. Suppose that \mathcal{O} is an absolutely unramified discrete valuation ring over $\mathbb{Z}_{(p)}$ of mixed characteristic $(0, p)$, with fraction field F . Then:*

- (1) $\text{disc}(L)$ is an \mathbb{F}_p -vector space of dimension at most 2.
- (2) $L_{\mathbb{Z}_p}$ is also maximal.
- (3) Suppose that we have an isometric embedding $(L, Q) \hookrightarrow (\tilde{L}, \tilde{Q})$ as a direct summand of a perfect lattice \tilde{L} . Set $\Lambda = L^\perp \subset \tilde{L}$. If $\Lambda' \subset \tilde{L}_{\mathcal{O}}$ is a sub-space isometric to $\Lambda_{\mathcal{O}}$, then one of the following statements holds:
 - (a) Λ' is a direct summand of $\tilde{L}_{\mathcal{O}}$.
 - (b) $(\Lambda')^\perp$ is a perfect lattice.

Proof. For (1), we first claim that $pL^\vee \subset L$. Otherwise, there exists $w \in L^\vee$ such that $p^2w \in L$ but $pw \notin L$. It is easy to see that $L + \langle pw \rangle \subset L^\vee$ is then a larger $\mathbb{Z}_{(p)}$ -lattice on which Q is $\mathbb{Z}_{(p)}$ -valued, giving us a contradiction. This shows that $\text{disc}(L)$ is an \mathbb{F}_p -vector space. To see that it has dimension at most 2, we note that the radical of $L_{\mathbb{F}_p}$ has dimension at most 2; cf. [Shi10, 29.10].

Now, note that $\text{disc}(L_{\mathbb{Z}_p}) = \text{disc}(L)$. Therefore, all \mathbb{Z}_p -lattices contained in $L_{\mathbb{Z}_p}^\vee$ and containing $L_{\mathbb{Z}_p}$ are already defined over $\mathbb{Z}_{(p)}$. This shows that $L_{\mathbb{Z}_p}$ is also maximal, giving us (2).

Let us now look at (3): First, note that Λ is also maximal. Indeed, for any direct summand $N \subset \tilde{L}$, both $\text{disc}(N)$ and $\text{disc}(N^\perp)$ are canonically isomorphic to $\tilde{L}/(N + N^\perp)$. In particular, since L is maximal, Λ has minimal possible discriminant among lattices in $\Lambda \otimes \mathbb{Q}$, and must also therefore be maximal. If $\Lambda' \subset \tilde{L}_{\mathcal{O}}$ is a direct summand, we are done. Otherwise, $(\Lambda')^{\perp\perp}$ is a direct summand of $\tilde{L}_{\mathcal{O}}$ properly containing Λ' and properly contained in $(\Lambda')^\vee$. But, by (1), $\text{disc}(\Lambda') \simeq \text{disc}(\Lambda)$ is an \mathbb{F}_p -vector space of dimension at most 2. It follows that $(\Lambda')^{\perp\perp}$, and therefore $(\Lambda')^\perp$, must be perfect lattices. \square

We now recall some definitions and results from [VZ10].

Definition 2.4. A $\mathbb{Z}_{(p)}$ -scheme X is **healthy regular** if it is regular, faithfully flat over $\mathbb{Z}_{(p)}$, and if, for every open sub-scheme $U \subset X$ containing $X_{\mathbb{Q}}$ and all generic points of $X_{\mathbb{F}_p}$, every abelian scheme over U extends uniquely to an abelian scheme over X .

A local $\mathbb{Z}_{(p)}$ -algebra R with maximal ideal \mathfrak{m} is **quasi-healthy regular** if it is regular, faithfully flat over $\mathbb{Z}_{(p)}$, and if every abelian scheme over $\mathrm{Spec} R \setminus \{\mathfrak{m}\}$ extends uniquely to an abelian scheme over $\mathrm{Spec} R$.

Theorem 2.5 (Vasiu-Zink). *Let R be a regular local $\mathbb{Z}_{(p)}$ -algebra with algebraically closed residue field k , of dimension at least 2, which admits a surjection*

$$R \twoheadrightarrow W(k)[[T_1, T_2]]/(p - h),$$

where $h \notin (p, T_1^p, T_2^p, T_1^{p-1}T_2^{p-1})$. Then R is quasi-healthy regular. In particular, if R is a smooth $\mathbb{Z}_{(p)}$ -algebra, then R is quasi-healthy regular.

Proof. This is [VZ10, Theorem 3]. □

Lemma 2.6. *Let (Z, q) be an anisotropic quadratic space over \mathbb{Q}_p . Then:*

- (1) $\dim Z \leq 4$.
- (2) *The sub-set*

$$M = \{z \in Z : q(z) \in \mathbb{Z}_p\}$$

is the unique maximal lattice in Z . In particular, for every unimodular element $m \in M$, $\nu_p(q(m)) \in \{0, 1\}$.

- (3) *Let $N \subset M_{\mathbb{F}_p}$ be the radical. Then*

$$\dim(N) \leq \min\{\dim Z, 2\}.$$

Proof. All these facts are very classical; cf. [Shi10, 25.5, 29.7, 29.10]. □

Let M_G^{loc} be the $\mathbb{Z}_{(p)}$ -scheme such that, for every $\mathbb{Z}_{(p)}$ -algebra R , we have:

$$M_G^{\mathrm{loc}}(R) = \{\text{Isotropic lines } L \subset L_R\}^2.$$

Proposition 2.7. *Let $N \subset L_{\mathbb{F}_p}$ be the radical, and let $r = \dim N$ and $s = t - r - 1$. We will assume that $N \neq L_{\mathbb{F}_p}$; or, equivalently, $s \geq 0$.*

- (1) M_G^{loc} is flat, projective of relative dimension $t - 2$ over $\mathbb{Z}_{(p)}$.
- (2) *The singular locus of $M_{G, \mathbb{F}_p}^{\mathrm{loc}}$ consists of lines contained in N , and so can be identified with $\mathbb{P}(N)$, and has co-dimension s in $M_{G, \mathbb{F}_p}^{\mathrm{loc}}$. In particular, if L is maximal, then it has co-dimension at least $t - 3$.³*
- (3) $M_{G, \mathbb{F}_p}^{\mathrm{loc}}$ is an lci variety. It is reduced if and only if $s \geq 1$. It is normal if and only if $s \geq 2$, and smooth if and only if $r = 0$.
- (4) *If L is maximal, then M_G^{loc} is healthy regular.*

Proof. (1) is a direct consequence of the hypothesis that $N \neq L_{\mathbb{F}_p}$. (2) is an easy fact about quadrics; the conclusion about the co-dimension of the singular locus when L is maximal follows from (2.6)(3). Finally, (3) follows from (2) and standard criteria for reducedness and normality.

To prove (4), we can and will work with $M_{G, \mathbb{Z}_p}^{\mathrm{loc}}$. We now invoke [Shi10, 29.8], where it is shown that there exists a Witt decomposition:

$$L_{\mathbb{Q}_p} = \bigoplus_{i=1}^{\nu} (\mathbb{Q}_p \cdot x_i \oplus \mathbb{Q}_p \cdot y_i) \oplus Z,$$

such that

$$L_{\mathbb{Z}_p} = \bigoplus_{i=1}^{\nu} (\mathbb{Z}_p \cdot x_i \oplus \mathbb{Z}_p \cdot y_i) \oplus M.$$

²A line is an R -sub-module that is locally a direct summand of rank 1.

³Here, $\mathbb{P}(N)$ denotes the space of lines in N .

Here, for each i , $\mathbb{Q}_p \cdot x_i \oplus \mathbb{Q}_p \cdot y_i$ is a copy of the hyperbolic plane, Z is anisotropic, and $M \subset Z$ is the unique maximal \mathbb{Z}_p -lattice. Note that $m = \dim Z$ is at most 4.

Suppose that the quadratic form on M , with respect to a suitable basis, is given by $\sum_{j=1}^m b_j T_j^2$. Then the projective co-ordinate ring for $M_{G, \mathbb{Z}_p}^{\text{loc}}$ is isomorphic to

$$\frac{\mathbb{Z}_p[U_i, L_i, T_j : 1 \leq i \leq \nu, 1 \leq j \leq m]}{(\sum_i U_i V_i + \sum_j b_j T_j^2)}.$$

We now show that the complete local rings of M_G^{loc} of dimension at least 2 are quasi-healthy regular. This will show (4).

Suppose first that we have a point $s \in M_G^{\text{loc}}(\nu)$ valued in a finite field k , where the homogeneous co-ordinates $U_1, \dots, U_\nu, V_1, \dots, V_\nu$ all vanish (otherwise, s is a smooth point, and the completion at s is quasi-healthy by (2.5)). We can assume that we have

$$[T_1(s) : T_2(s) : \dots : T_m(s)] = [1 : a_2 : \dots : a_m],$$

for $a_i \in k$ (Note that $m \geq 1$, necessarily). If we denote the Teichmüller representative of a_i in $W(k)$ again by a_i , the complete local ring of M_G^{loc} at s is

$$\hat{\mathcal{O}}_s = \frac{W(k)[u_i, v_i, t_j : 1 \leq i \leq \nu; 2 \leq j \leq m]}{(\sum_i u_i v_i + \sum_{j \geq 2} b_j t_j^2 + 2 \sum_{j \geq 2} a_j b_j t_j + b_1 + \sum_{j \geq 2} b_j a_j^2)}.$$

We first claim that

$$(2.7.1) \quad \nu_p \left(b_1 + \sum_{j \geq 2} b_j a_j^2 \right) = 1.$$

This follows from (2.6)(2). In particular, we find that $\hat{\mathcal{O}}_s$ must be regular.

If $\nu \geq 1$, it is easy to see that we have a surjective map

$$\hat{\mathcal{O}}_s \twoheadrightarrow W(k)[[X_1, X_2]]/(p - X_1 X_2).$$

So $\hat{\mathcal{O}}_s$ is quasi-healthy regular, by (2.5).

If $\nu = 0$, then we must have $m = \dim Z \geq 3$. If b_1 is a unit, then by (2.7.1) at least one of the b_j , for $j \geq 2$, must also be a unit. So, without loss of generality, we can assume that b_2 is a unit. Given this, it is simple to find a surjective map

$$\hat{\mathcal{O}}_s \twoheadrightarrow W(k)[[X_1, X_2]]/(p - \alpha_1 X_1^2 - \alpha_2 X_2^2),$$

where at least one of the α_i is a unit in $W(k)$. Again, by (2.5), we conclude that $\hat{\mathcal{O}}_s$ is quasi-healthy regular.

If $\dim N = 1$, so that the singular locus is a point, all the complete local rings of M_G^{loc} at the non-closed points are formally smooth over \mathbb{Z}_p and are therefore quasi-healthy regular. If $t = 3$, then the only points whose complete local rings have dimension at least 2 are the closed points, and so we are again done.

We are therefore left with the case where $\dim N = 2$ and $t \geq 4$. In this case, the singular locus is a rational curve in the special fiber, and we have to investigate the complete local ring $\hat{\mathcal{O}}_\eta$ of M_G^{loc} at the generic point η of this curve. Since $\dim N = 2$, we can assume that

$$\nu_p(b_j) = \begin{cases} 1 & \text{if } j = 1, 2; \\ 0 & \text{otherwise.} \end{cases}$$

We then find that $\hat{\mathcal{O}}_\eta$ is isomorphic to the completion of the ring

$$\frac{\mathbb{Z}_p[u_i, v_i, t_j : 1 \leq i \leq \nu; 2 \leq j \leq m]}{(\sum_i u_i v_i + b_1 + \sum_{j \geq 2} b_j t_j^2)}$$

at the ideal $(p, u_i, v_i, t_j : 1 \leq i \leq \nu; 3 \leq j \leq m)$. In particular, the residue field of $\hat{\mathcal{O}}_\eta$ is $k_p(t_2)$. Let k be an algebraic closure of $k_p(t_2)$; then there exists a faithfully flat local $\hat{\mathcal{O}}_\eta$ -algebra R of the form

$$R = \frac{W(k)[[u_i, v_i, t_j : 1 \leq i \leq r; 3 \leq j \leq m]]}{(\sum_i u_i v_i + \sum_{j \geq 3} b_j t_j^2 + p\alpha)},$$

where $\alpha \in W(k)^\times$. Now, the same argument as above, considering the cases $\nu \neq 0$ and $\nu = 0$ separately, shows that R is quasi-healthy regular. This implies that $\hat{\mathcal{O}}_\eta$ is also quasi-healthy regular and completes the proof of (4). \square

3. SPIN GROUP SHIMURA VARIETIES

Fix a prime $p > 2$, and let (L, Q) be a quadratic space over $\mathbb{Z}_{(p)}$ of signature $(n, 2)$ with $n \geq 1$. By this, we mean that the largest positive definite sub-space of $L_\mathbb{R}$ has dimension n . Set $V = L_\mathbb{Q}$. Let G be the smooth $\mathbb{Z}_{(p)}$ -group scheme attached to L in (2.1), so that $G_\mathbb{Q} = \mathrm{GSpin}(V, Q)$.

3.1. Let X be the space of oriented negative definite 2-planes in $V_\mathbb{R}$. The points of X correspond to certain Hodge structures of weight 0 on the vector space V , polarized by Q : Fix $\mathbf{h} \in X$, and suppose that $(e_\mathbf{h}, f_\mathbf{h})$ is an oriented, orthogonal basis for the oriented negative definite 2-plane attached to \mathbf{h} , with $Q(e_\mathbf{h}) = Q(f_\mathbf{h}) = -1$. Also fix a square root of -1 , $\sqrt{-1} \in \mathbb{C}$. Set

$$L_\mathbf{h}^{p,q} = \begin{cases} \langle e_\mathbf{h} + \sqrt{-1}f_\mathbf{h} \rangle \subset V_\mathbb{C} & \text{if } (p, q) = (-1, 1); \\ \langle e_\mathbf{h}, f_\mathbf{h} \rangle^\perp \subset V_\mathbb{C} & \text{if } (p, q) = (0, 0); \\ \langle e_\mathbf{h} - \sqrt{-1}f_\mathbf{h} \rangle \subset V_\mathbb{C} & \text{if } (p, q) = (1, -1); \\ 0 & \text{otherwise.} \end{cases}$$

Then $L_\mathbf{h}$ is a $\mathbb{Z}_{(p)}$ -Hodge structure of weight 0 with underlying $\mathbb{Z}_{(p)}$ -module L ; the associated \mathbb{Q} -Hodge structure is polarized by Q . We also find that X admits a natural embedding as an open sub-space of $\mathbb{P}(V_\mathbb{C})$ and that its two connected components are switched by complex conjugation.

Lemma 3.2. *The pair $(G_\mathbb{Q}, X)$ is a Shimura datum with reflex field \mathbb{Q} .*

Proof. We attach to $\mathbf{h} \in X$ the map $\mathbb{C} \rightarrow C_\mathbb{R}$ that carries $\sqrt{-1}$ to $e_\mathbf{h}f_\mathbf{h}$. This in turn allows us to attach to \mathbf{h} a co-character $j_\mathbf{h} : \mathbb{S} \rightarrow G_\mathbb{R}$, and one easily checks that this makes $(G_\mathbb{Q}, X)$ a Shimura datum. The key point is that the conjugation action of $z_\mathbf{h} = e_\mathbf{h}f_\mathbf{h}$ on $V_\mathbb{R}$ is simply reflection in the plane $\langle e_\mathbf{h}, f_\mathbf{h} \rangle$, and so the form $[v_1, z_\mathbf{h}v_2]_Q$ is positive definite on $V_\mathbb{R}$.

To compute the reflex field, choose \mathbf{h} such that $e_\mathbf{h} = a \cdot e$ and $f_\mathbf{h} = b \cdot f$, for $e, f \in V$ and $a, b \in \mathbb{R}_{>0}$. Suppose that $\alpha = Q(e)$ and $\beta = Q(f)$, and let $E = \mathbb{Q}(\sqrt{\alpha\beta})$; then $j_\mathbf{h}$ descends to a map $\mathrm{Res}_{E/\mathbb{Q}} G_m \rightarrow G_E$, and so the reflex field of $(G_\mathbb{Q}, X)$ is contained in E . Since $n \geq 1$, there exists $u \in \langle e, f \rangle^\perp \subset V$ such that $Q(u) = \gamma > 0$. Replacing e with $e + ku$ for appropriate $k \in \mathbb{Q}$, shows that $E(G_\mathbb{Q}, X)$ is contained in $\mathbb{Q}(\sqrt{(\alpha + k^2\gamma)\beta})$ for all k of sufficiently small magnitude. Therefore, we must have $E(G_\mathbb{Q}, X) = \mathbb{Q}$. \square

3.3. Fix a compact open sub-group $K \subset G(\mathbb{A}_f)$, and let $\mathrm{Sh}_K(G_\mathbb{Q}, X)$ be the associated Shimura variety over \mathbb{Q} . We will assume that K is of the form $K_p K^p$, where $K_p = G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$. We will also assume that K^p is chosen to be small enough, so that $\mathrm{Sh}_K(G_\mathbb{Q}, X)$ is an honest smooth variety and not just an algebraic space. By weak approximation for $G_\mathbb{Q}$ (which can be deduced from weak approximation for its derived group, which is simply connected [PR94, Theorem 7.8]), we have an identification of complex analytic varieties:

$$\mathrm{Sh}_K(G_\mathbb{Q}, X)_\mathbb{C}^{\mathrm{an}} = G(\mathbb{Z}_{(p)}) \backslash (X \times G(\mathbb{A}_f^p) / K^p).$$

Viewing L as a representation of the discrete group $G(\mathbb{Z}_{(p)})$, we obtain a $\mathbb{Z}_{(p)}$ -local system \mathbf{L}_B over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}^{\mathrm{an}}$. Also, X is an analytic open sub-variety of the complex quadric \check{X} that parameterizes isotropic lines in $V_{\mathbb{C}}$. As such, there exists a tautological $G(\mathbb{R})$ -equivariant filtered vector bundle $(V \otimes \mathcal{O}_X^{\mathrm{an}}, F^{\bullet}(V \otimes \mathcal{O}_X^{\mathrm{an}}))$ with

$$\begin{aligned} F^2(V \otimes \mathcal{O}_X^{\mathrm{an}}) &= 0, \\ F^1(V \otimes \mathcal{O}_X^{\mathrm{an}}) &\subset V \otimes \mathcal{O}_X^{\mathrm{an}} \text{ is the tautological isotropic line,} \\ F^0(V \otimes \mathcal{O}_X^{\mathrm{an}}) &= F^1(V \otimes \mathcal{O}_X^{\mathrm{an}})^{\perp}, \\ F^{-1}(V \otimes \mathcal{O}_X^{\mathrm{an}}) &= V \otimes \mathcal{O}_X^{\mathrm{an}}. \end{aligned}$$

In fact, the specialization of this filtered vector bundle at any point $\mathbf{h} \in X$ gives us the Hodge structure $V_{\mathbf{h}}$. Since it is $G_{\mathbb{R}}$ -equivariant, it descends to a filtered analytic vector bundle $(V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}}, F^{\bullet}V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}})$ over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}^{\mathrm{an}}$. Moreover, this vector bundle is equipped with a natural integrable connection with respect to which $F^{\bullet}V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}}$ satisfies Griffiths transversality, and for which the horizontal sections are canonically identified with the local system $\mathbf{L}_B \otimes \mathbb{C}$. In sum, the tuple $(\mathbf{L}_B, V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}}, F^{\bullet}V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}})$ is a variation of $\mathbb{Z}_{(p)}$ -Hodge structures on $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}^{\mathrm{an}}$. The associated variation of \mathbb{Q} -Hodge structures is polarized by the quadratic form on \mathbf{L}_B .

There exists a smooth toroidal compactification of $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}$ such that $V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}}$ has regular singular points along the boundary divisor (cf. [AMRT10, Har89]). So, by the main result of [Del70], $V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}}$ algebraizes to an algebraic vector bundle with integrable connection $V_{\mathrm{dR}, \mathbb{C}}$. Moreover, the filtration $F^{\bullet}V_{\mathrm{dR}, \mathbb{C}}^{\mathrm{an}}$ is, Zariski locally on $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}$, associated with a map to the projective variety \check{X} , and so also algebraizes to a filtration $F^{\bullet}V_{\mathrm{dR}, \mathbb{C}}$. We will see below that $(V_{\mathrm{dR}, \mathbb{C}}, F^{\bullet}V_{\mathrm{dR}, \mathbb{C}})$ descends to a filtered vector bundle over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ with integrable connection.

For any $\mathbb{Z}_{(p)}$ -algebra R , we can form the R -local system $\mathbf{L}_B \otimes R$. If $R = \mathbb{Z}_p, \mathbb{Q}_{\ell}, \mathbb{A}_f^p$, then $\mathbf{L}_B \otimes R$ is in fact an R -local system on the algebraic variety $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{Q}}$. We will denote the associated local systems by $\mathbf{L}_p, \mathbf{V}_{\ell}$ and $\mathbf{V}_{\mathbb{A}_f^p}$, respectively. We will see below that all these sheaves descend to $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$.

3.4. Let $C = C(L, Q)$ be the Clifford algebra for (L, Q) . As usual, when viewing it as a representation of G , we denote it by the letter H . For $\delta \in C \cap C_{\mathbb{Q}}^{\times}$ satisfying $\delta^* = -\delta$, set $\mathcal{G}_{\delta, \mathbb{Q}} = \mathrm{GSp}(H_{\mathbb{Q}}, \psi_{\delta})$, so that we have the Kuga-Satake embedding $\check{G}_{\mathbb{Q}} \hookrightarrow \mathcal{G}_{\delta, \mathbb{Q}}$. Let \mathcal{X} be the space of Lagrangian sub-spaces $W \subset H_{\mathbb{C}}$ (with respect to the form ψ_{δ}) such that the Hermitian form $\sqrt{-1}\psi_{\delta}(w_1, \bar{w}_2)$ restricts to a (positive or negative) definite form on W : this is simply the union of the Siegel half-spaces attached to (H, ψ_{δ}) .

Lemma 3.5. *One can choose δ so that the Kuga-Satake embedding induces an embedding of Shimura data $(G_{\mathbb{Q}}, X) \hookrightarrow (\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X})$.*

Proof. Let $e_{\mathbf{h}}, f_{\mathbf{h}} \in V_{\mathbb{R}}$ be as in the proof of (3.2), so that there exist $e, f \in L$ with $e_{\mathbf{h}} = ae$ and $f_{\mathbf{h}} = bf$, and take $\delta = fe$. By the recipe in (1.14), the associated Lagrangian sub-space of $H_{\mathbb{C}}$ is $W_{\mathbf{h}} = \mathrm{im}(e_{\mathbf{h}} - \sqrt{-1}f_{\mathbf{h}})$. Finding δ such that $\sqrt{-1}\psi_{\delta}(z_1, \bar{z}_2)$ is definite on $W_{\mathbf{h}}$ amounts to finding δ such that the symmetric form

$$(z_1, z_2) \mapsto \psi_{\delta}(efz_1, z_2)$$

is definite on L . We claim that $\delta = fe$ works. To see this, we only have to observe that $\langle e, f \rangle^{\perp} \subset L$ is a positive definite quadratic space, and that:

$$efvfe = \begin{cases} v & \text{if } v \in \langle e, f \rangle^{\perp}; \\ -v & \text{if } v \in \langle e, f \rangle. \end{cases}$$

□

3.6. Let $\mathcal{K}_p \subset \mathcal{G}_{\delta, \mathbb{Q}}(\mathbb{Q}_p)$ be the stabilizer of $H_{\mathbb{Z}_p}$. Then we have $K_p \subset \mathcal{K}_p \cap G(\mathbb{Q}_p)$. Let $K = K_p K^p \subset G(\mathbb{A}_f)$ be as above; then, for any compact open $\mathcal{K}^p \subset \mathcal{G}_{\delta}(\mathbb{A}_f^p)$ containing K_p , we have a finite, unramified map of canonical models of Shimura varieties over \mathbb{Q} :

$$\mathrm{Sh}_K(G_{\mathbb{Q}}, X) \rightarrow \mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X}).$$

Here, $\mathcal{K} = \mathcal{K}_p \mathcal{K}^p$. We will call this a **Kuga-Satake map**.

$\mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X})$ has a natural moduli description, which allows us to construct an integral canonical model $\mathcal{S}_{\mathcal{K}}$ for $\mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X})$ over $\mathbb{Z}_{(p)}$. To describe this, we will work with abelian schemes up to prime-to- p isogeny. More precisely, given a scheme T , the category $AV_{(p)}(T)$ of abelian schemes up to prime-to- p isogeny has for its objects abelian schemes A over T , where, for two abelian schemes A and B over T , the space of morphisms from A to B is the $\mathbb{Z}_{(p)}$ -module

$$\mathrm{Hom}(A, B)_{(p)} = \mathrm{Hom}(A, B) \otimes \mathbb{Z}_{(p)}.$$

Given an abelian scheme A over T , a **quasi-polarization** (or simply **polarization**) of A in $AV_{(p)}(T)$ will be an element $\lambda \in \mathrm{Hom}(A, A^{\vee})_{(p)}$ that is a positive multiple of a polarization $\lambda' : A \rightarrow A^{\vee}$.

From now on we will suppress the qualifying phrase ‘up to prime-to- p isogeny’: all abelian schemes will only be considered in the prime-to- p isogeny category.

Given any scheme T , a prime ℓ invertible in T , and an abelian scheme $f : A \rightarrow T$, we can consider the associated relative first ℓ -adic cohomology sheaf $R^1 f_* \underline{\mathbb{Q}}_{\ell}$. Let $\underline{\mathbb{Q}}_{\ell}(-1)$ be the Tate twist: it is the relative first ℓ -adic cohomology of $\mathbb{G}_{m, T}$ over T .

If T is a $\mathbb{Z}_{(p)}$ -scheme, then we can make sense of the restricted products:

$$\underline{\mathbb{A}}_f^p(-1) = \prod'_{\ell \neq p} \underline{\mathbb{Q}}_{\ell}(-1) \quad ; \quad R^1 f_* \underline{\mathbb{A}}_f^p = \prod'_{\ell \neq p} R^1 f_* \underline{\mathbb{Q}}_{\ell}$$

as étale sheaves over T . Given a polarization $\lambda : A \rightarrow A^{\vee}$, we get an induced non-degenerate Poincaré pairing of $\underline{\mathbb{A}}_f^p$ -sheaves

$$\psi_{\lambda} : R^1 f_* \underline{\mathbb{A}}_f^p \otimes R^1 f_* \underline{\mathbb{A}}_f^p \rightarrow \underline{\mathbb{A}}_f^p(-1).$$

Suppose that we are given $\lambda : A \rightarrow A^{\vee}$ and an isomorphism

$$\eta : \underline{H \otimes \mathbb{A}}_f^p \xrightarrow{\sim} R^1 f_* \underline{\mathbb{A}}_f^p$$

of $\underline{\mathbb{A}}_f^p$ -sheaves over T . Then we obtain two different non-degenerate pairings on $\underline{H \otimes \mathbb{A}}_f^p$: The first is the constant pairing into $\underline{\mathbb{A}}_f^p$ arising from ψ_{δ} , which we will again call ψ_{δ} ; and the second is the pairing $\eta^* \psi_{\lambda}$ into $\underline{\mathbb{A}}_f^p(-1)$ obtained by pulling back ψ_{λ} along η . In particular, the existence of η implies that $\underline{\mathbb{A}}_f^p(-1)$ is trivializable over T . We say that η **respects polarizations** if, for some choice of isomorphism $\underline{\mathbb{A}}_f^p(-1) \xrightarrow{\sim} \underline{\mathbb{A}}_f^p$, the pairings $\eta^* \psi_{\lambda}$ and ψ_{δ} agree.

In the above situation, we can consider the étale sheaf $I^p(A, \lambda)$ over T defined as follows: For every T -scheme Z , $I^p(A, \lambda)(Z)$ will be the set of polarization preserving isomorphisms

$$\eta : (\underline{H \otimes \mathbb{A}}_f^p)_Z \xrightarrow{\sim} (R^1 f_* \underline{\mathbb{A}}_f^p)_Z.$$

Then $I^p(A, \lambda)$ naturally has an action by $\mathcal{G}_{\delta}(\mathbb{A}_f^p)$ on the right.

For any $\mathbb{Z}_{(p)}$ -scheme T , let $\mathcal{S}_{\mathcal{K}}(T)$ be the set of isomorphism classes of tuples $(A, \lambda, [\eta])$, where:

- (A, λ) is a polarized abelian scheme over T .
- $[\eta]$ is a **\mathcal{K}^p -level structure**: it is a section of the quotient sheaf $I^p(A, \lambda)/\mathcal{K}^p$.

For \mathcal{K} sufficiently small, $\mathcal{S}_{\mathcal{K}}$ is (represented by) a quasi-projective scheme over $\mathbb{Z}_{(p)}$, whose generic fiber is canonically identified with $\mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X})$.

3.7. We return now to the Kuga-Satake map

$$\mathrm{Sh}_K(G_{\mathbb{Q}}, X) \rightarrow \mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X}).$$

We assume that K^p and \mathcal{K}^p are chosen such that $\mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_{\delta, \mathbb{Q}}, \mathcal{X})$ admits the above description as a fine moduli scheme over \mathbb{Q} with integral model $\mathcal{S}_{\mathcal{K}}$ over $\mathbb{Z}_{(p)}$.

Let \mathcal{S}_K be the normalization of $\mathcal{S}_{\mathcal{K}}$ in $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$: all we know about it for now is that it is a normal, flat $\mathbb{Z}_{(p)}$ -scheme. The following result is one we would like to (suitably) generalize in this paper:

Theorem 3.8 (Kisin, [Kis10, 2.3.8]). *Suppose that (L, Q) is a perfect quadratic space over $\mathbb{Z}_{(p)}$. Then \mathcal{S}_K is smooth over $\mathbb{Z}_{(p)}$.*

3.9. Over $\mathcal{S}_{\mathcal{K}}$, we have the tautological tuple $(A, \lambda, [\eta])$. Let $(A^{\mathrm{KS}}, \lambda^{\mathrm{KS}}, [\eta^{\mathrm{KS}}])$ be the induced tuple over \mathcal{S}_K : this is the **Kuga-Satake abelian scheme** over \mathcal{S}_K . Let $A_{\mathbb{Q}}^{\mathrm{KS}}$ denote the induced abelian scheme over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$. The characteristic 0 theory of Shimura varieties (cf. the argument in (6.5)) shows that we have a natural map of $\mathbb{Z}_{(p)}$ -algebras

$$\rho^{\mathrm{KS}} : C \rightarrow \mathrm{End}(A_{\mathbb{Q}}^{\mathrm{KS}})_{(p)}.$$

By the theory of Néron models (cf. [FC90, I.2.7]), when (L, Q) is perfect, it follows from (3.8) that the map in fact factors through $\mathrm{End}(A^{\mathrm{KS}})_{(p)}$. Moreover, the grading on H gives rise to a $\mathbb{Z}/2\mathbb{Z}$ -grading $A^{\mathrm{KS}} = A^{\mathrm{KS}, +} \times A^{\mathrm{KS}, -}$ that is compatible with the grading on C .

We will now use the Kuga-Satake embedding to give a motivic interpretation of the sheaves $\mathbf{L}_B, \mathbf{L}_p, \mathbf{V}_{\mathrm{dR}, \mathbb{C}}, \mathbf{V}_{\ell}$ encountered above.

3.10. Let \mathbf{H}_B be the first relative Betti cohomology with coefficients in $\mathbb{Z}_{(p)}$ of the analytification of $A_{\mathbb{C}}^{\mathrm{KS}}$ over $\mathcal{S}_{K, \mathbb{C}}^{\mathrm{an}}$. It is the sheaf attached to the G -representation W via the construction in (3.1). The tensor $\pi \in H_{\mathbb{Q}}^{\otimes(2,2)}$ gives rise to a global section

$$\pi_B \in H^0(\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}^{\mathrm{an}}, \mathbf{H}_B^{\otimes(2,2)} \otimes \mathbb{Q}).$$

We can view π_B as an idempotent endomorphism of the local system $\mathbf{H}_B^{\otimes(1,1)} \otimes \mathbb{Q}$, and we can identify the image of π_B with the local system $\mathbf{L}_B \otimes \mathbb{Q}$. In fact, it follows from (1.7) that the image of $\mathbf{H}_B^{\otimes(2,2)}$ under π_B is the local system \mathbf{L}_B^{\vee} attached to the dual lattice $L^{\vee} \subset L$. This allows us to recover \mathbf{L}_B as well: it is the $\mathbb{Z}_{(p)}$ -sub-local system of $\mathbf{L}_B \otimes \mathbb{Q}$ that pairs integrally with \mathbf{L}_B^{\vee} .

For every point $s \in \mathcal{S}_K(\mathbb{C})$, we can identify the fiber $\mathbf{H}_{B,s}^{\otimes(1,1)}$ with the $\mathbb{Z}_{(p)}$ -module $\mathrm{End}(H_B^1(A_s^{\mathrm{KS}}, \mathbb{Z}_{(p)}))$, and so we can view $\mathbf{L}_{B,s}$ as a space of C -equivariant endomorphisms of $H_B^1(A_s^{\mathrm{KS}}, \mathbb{Z}_{(p)})$. Just as in (1.5), the form

$$[\varphi_1, \varphi_2] = \frac{1}{2^{n+2}} \mathrm{Tr}(\varphi_1 \circ \varphi_2)$$

equips $\mathrm{End}(H_B^1(A_s^{\mathrm{KS}}, \mathbb{Z}_{(p)}))$ with a bi-linear form, restricting to the quadratic form on $\mathbf{L}_{B,s}$. We note that $\pi_{B,s}$ is in fact an endomorphism of Hodge structures, and so $\mathbf{L}_{B,s}$ is a Hodge sub-structure of $\mathrm{End}(H_B^1(A_s^{\mathrm{KS}}, \mathbb{Z}_{(p)}))$.

3.11. For $\ell \neq p$, let \mathbf{H}_{ℓ} denote the first relative ℓ -adic cohomology sheaf of A^{KS} over \mathcal{S}_K : by this we mean that we are taking our coefficients to be $\underline{\mathbb{Q}}_{\ell}$. We will also consider the restricted product

$$\mathbf{H}_{\mathbb{A}_f^p} = \prod_{\ell \neq p}^{\prime} \mathbf{H}_{\ell}.$$

Note that, for any geometric point $s \rightarrow \mathcal{S}_K$, we have

$$\mathbf{H}_{\ell,s} = H_{\mathrm{ét}}^1(A_s^{\mathrm{KS}}, \mathbb{Q}_{\ell}).$$

By [Kis10, 2.2.1], for each prime $\ell \neq p$, we obtain a global section

$$\pi_\ell \in H^0(\mathrm{Sh}_K(G_{\mathbb{Q}}, X), \mathbf{H}_\ell^{\otimes(2,2)}).$$

As in the case of Betti cohomology, we can view π_ℓ as an endomorphism of the ℓ -adic sheaf $\mathbf{H}_\ell^{\otimes(1,1)}$. Let $\mathbf{V}_\ell \subset \mathbf{H}_\ell^{\otimes(1,1)}$ be the image of π_ℓ . This conflation of notation is no co-incidence, since this construction recovers the ℓ -adic sheaf \mathbf{V}_ℓ already defined in (3.1) over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}$.

Given a geometric point $s \rightarrow \mathrm{Sh}_K(G_{\mathbb{Q}}, X)$, $\mathbf{H}_{\ell,s}^{\otimes(1,1)}$ can be identified with $\mathrm{End}(H_{\mathrm{et}}^1(A_s^{\mathrm{KS}}, \mathbb{Q}_\ell))$, and so we can view the fiber $\mathbf{V}_{\ell,s}$ as a space of endomorphisms of the \mathbb{Q}_ℓ -vector space $H_{\mathrm{et}}^1(A_s^{\mathrm{KS}}, \mathbb{Q}_\ell)$.

Assume now that (L, Q) is perfect. Then, since \mathcal{S}_K is smooth, π_ℓ is in fact a section of $\mathbf{H}_\ell^{\otimes(2,2)}$ over \mathcal{S}_K , and so \mathbf{V}_ℓ extends to a sheaf over \mathcal{S}_K . Let I_G^p be the sub-sheaf of $I^p(A^{\mathrm{KS}}, \lambda^{\mathrm{KS}})$ consisting of graded isomorphisms

$$\eta : H \otimes \mathbb{A}_f^p \xrightarrow{\sim} \mathbf{H}_{\mathbb{A}_f^p}$$

that are C -equivariant and that carry π to $\pi_{\mathbb{A}_f^p}$, where $\pi_{\mathbb{A}_f^p}$ is the obvious product of the π_ℓ , for $\ell \neq p$. Note that $G(\mathbb{A}_f^p)$ naturally acts on the right on I_G^p via pre-composition, and that we have a natural map of quotient sheaves

$$I_G^p/K^p \rightarrow I^p(A^{\mathrm{KS}}, \lambda^{\mathrm{KS}})/K^p.$$

Proposition 3.12. *There is a canonical K^p -level structure $[\eta_G^{\mathrm{KS}}]$ on A^{KS} ; that is, a section of I_G^p/K^p over \mathcal{S}_K , such that $[\eta^{\mathrm{KS}}]$ is its image in $I^p(A^{\mathrm{KS}}, \lambda^{\mathrm{KS}})/K^p$. Therefore, for every geometric point $s \rightarrow \mathcal{S}_K$, there exists a canonical $\pi_1(\mathcal{S}_K, s)$ -stable K^p -orbit of C -equivariant graded isomorphisms*

$$\eta : H \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_{\mathrm{et}}^1(A_s^{\mathrm{KS}}, \mathbb{A}_f^p)$$

carrying π to $\pi_{\mathbb{A}_f^p,s}$. In particular, there exists a canonical $\pi_1(\mathcal{S}_K, s)$ -stable K^p -orbit of isometries:

$$\eta : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} \mathbf{V}_{\mathbb{A}_f^p,s}.$$

Remark 3.13. Even if (V, Q) is not perfect, such a canonical K^p -level structure always exists over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$.

3.14. When $\ell = p$, we will permit a little inconsistency in our notation, and denote by \mathbf{H}_p the first relative étale cohomology of $A_{\mathbb{Q}}^{\mathrm{KS}}$ over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ with coefficients in \mathbb{Z}_p (as opposed to \mathbb{Q}_p). Its fiber at any geometric point $s \rightarrow \mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ is $H_{\mathrm{et}}^1(A_s^{\mathrm{KS}}, \mathbb{Z}_p)$.

The projector π induces a projector π_p on $\mathbf{H}_p^{\otimes(1,1)} \otimes \mathbb{Q}_p$. Its image is a descent over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ of the p -adic sheaf $\mathbf{L}_p \otimes \mathbb{Q}_p$. Just as in (3.10), we can also use π_p to descend \mathbf{L}_p and its dual \mathbf{L}_p^\vee over $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$. For every geometric point $s \rightarrow \mathrm{Sh}_K(G_{\mathbb{Q}}, X)$, there exists a C -equivariant graded isomorphism

$$H \otimes \mathbb{Z}_p \xrightarrow{\sim} H_{\mathrm{et}}^1(A_s^{\mathrm{KS}}, \mathbb{Z}_p)$$

carrying π to $\pi_{p,s}$. In particular, there exists an isometry

$$L \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathbf{L}_{p,s}$$

of quadratic spaces over \mathbb{Z}_p .

3.15. Let \mathbf{H}_{dR} be the first relative de Rham cohomology of A^{KS} over \mathcal{S}_K , equipped with its Hodge filtration and the Gauss-Manin connection. Let $\mathbf{H}_{\mathrm{dR},\mathbb{Q}}$ be its restriction to $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$. As shown in [Kis10, 2.2.2], π gives rise to a global section

$$\pi_{\mathrm{dR},\mathbb{Q}} \in H^0(\mathrm{Sh}_K(G_{\mathbb{Q}}, X), \mathbf{H}_{\mathrm{dR},\mathbb{Q}}^{\otimes(2,2)})$$

that is parallel for the Gauss-Manin connection and lies in F^0 for the Hodge filtration. Let $\mathbf{V}_{\mathrm{dR}} \subset \mathbf{H}_{\mathrm{dR},\mathbb{Q}}^{\otimes(1,1)}$ be the image of $\pi_{\mathrm{dR},\mathbb{Q}}$; here, we are viewing $\pi_{\mathrm{dR},\mathbb{Q}}$ as an endomorphism of the vector bundle $\mathbf{H}_{\mathrm{dR},\mathbb{Q}}^{\otimes(1,1)}$, equipped with its integrable connection. Since $\pi_{\mathrm{dR},\mathbb{Q}}$ is parallel, \mathbf{V}_{dR} inherits a Gauss-Manin connection. It also inherits a filtration $F^\bullet \mathbf{V}_{\mathrm{dR}}$ from the Hodge filtration on $\mathbf{H}_{\mathrm{dR},\mathbb{Q}}^{\otimes(2,2)}$. In fact, the filtered vector bundle $(\mathbf{V}_{\mathrm{dR}}, F^\bullet \mathbf{V}_{\mathrm{dR}})$ with integrable connection is a descent over $\mathrm{Sh}_K(G_{\mathbb{Q}}, K)$ of the filtered vector bundle $(\mathbf{V}_{\mathrm{dR},\mathbb{C}}, F^\bullet \mathbf{V}_{\mathrm{dR},\mathbb{C}})$ considered in (3.1).

For any point $s \rightarrow \mathrm{Sh}_K(G_{\mathbb{Q}}, X)$, we can identify $\mathbf{H}_{\mathrm{dR},s}^{\otimes(1,1)}$ with $\mathrm{End}(H_{\mathrm{dR}}^1(A_s^{\mathrm{KS}}/k(s)))$, and so we can view $\mathbf{V}_{\mathrm{dR},s}$ as a space of endomorphisms of the $k(s)$ -vector space $H_{\mathrm{dR}}^1(A_s^{\mathrm{KS}}/k(s))$. As above, we can equip $\mathbf{V}_{\mathrm{dR},s}$ with a quadratic form $\mathbf{Q}_{\mathrm{dR},s}$. In fact, there exists a global quadratic form $\mathbf{Q}_{\mathrm{dR},\mathbb{Q}}$ on \mathbf{V}_{dR} with values in $\mathcal{O}_{\mathrm{Sh}_K(G_{\mathbb{Q}}, X)}$ that is parallel for the Gauss-Manin connection and that specializes to the quadratic form on $\mathbf{V}_{\mathrm{dR},s}$ at each point s .

Proposition 3.16. *Suppose that (L, Q) is perfect. Then:*

- (1) $\pi_{\mathrm{dR},\mathbb{Q}}$ extends to a section π_{dR} of $\mathbf{H}_{\mathrm{dR}}^{\otimes(2,2)}$ over \mathcal{S}_K , and so $(\mathbf{V}_{\mathrm{dR}}, F^\bullet \mathbf{V}_{\mathrm{dR}})$ extends to a filtered vector bundle $(\mathbf{L}_{\mathrm{dR}}, F^\bullet \mathbf{L}_{\mathrm{dR}})$ over \mathcal{S}_K equipped with an integrable connection. The quadratic form $\mathbf{Q}_{\mathrm{dR},\mathbb{Q}}$ extends to a quadratic form \mathbf{Q}_{dR} on \mathbf{L}_{dR} .

- (2) Consider the functor $\mathcal{P}_{\mathrm{dR}}$ on \mathcal{S}_K -schemes given by

$$\mathcal{P}_{\mathrm{dR}}(T) = \left(\begin{array}{c} C\text{-equivariant graded } \mathcal{O}_T\text{-module isomorphisms} \\ \xi : H \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_T \xrightarrow{\sim} \mathbf{H}_{\mathrm{dR},T} \\ \text{that carry } \pi \text{ to } \pi_{\mathrm{dR}}. \end{array} \right)$$

Then $\mathcal{P}_{\mathrm{dR}}$ is a G -torsor over \mathcal{S}_K . In particular, fppf (in fact, étale) locally on \mathcal{S}_K , \mathbf{L}_{dR} is isometric to $L \otimes \mathcal{O}_{\mathcal{S}_K}$.

- (3) Let $G_{\mathrm{dR}} \subset \mathrm{GL}^+(\mathbf{H}_{\mathrm{dR}})$ be the stabilizer of π_{dR} within the commutant of C . Then $G_{\mathrm{dR}} = \mathrm{GSpin}(\mathbf{L}_{\mathrm{dR}}, \mathbf{Q}_{\mathrm{dR}})$ is the inner twist of G by the torsor $\mathcal{P}_{\mathrm{dR}}$.
- (4) The Hodge filtration $F^\bullet \mathbf{H}_{\mathrm{dR}}$ is G_{dR} -split (cf. [Kis10, 1.1.2]). More precisely, the filtration $F^\bullet \mathbf{L}_{\mathrm{dR}}$ is a three-step filtration

$$0 = F^2 \mathbf{L}_{\mathrm{dR}} \subset F^1 \mathbf{L}_{\mathrm{dR}} \subset F^0 \mathbf{L}_{\mathrm{dR}} = (F^1 \mathbf{L}_{\mathrm{dR}})^\perp \subset F^{-1} \mathbf{L}_{\mathrm{dR}} = \mathbf{L}_{\mathrm{dR}},$$

with $F^1 \mathbf{L}_{\mathrm{dR}} \subset \mathbf{L}_{\mathrm{dR}}$ an isotropic line, and we have:

$$F^1 \mathbf{H}_{\mathrm{dR}} = \ker(F^1 \mathbf{L}_{\mathrm{dR}}) = \mathrm{im}(F^1 \mathbf{L}_{\mathrm{dR}}).$$

In (4), we are viewing \mathbf{L}_{dR} as a space of endomorphisms for \mathbf{H}_{dR} , and by $\ker(F^1 \mathbf{L}_{\mathrm{dR}})$ we mean the sub-space of $F^1 \mathbf{H}_{\mathrm{dR}}$ that is locally annihilated by any generating section of $F^1 \mathbf{L}_{\mathrm{dR}}$. Similarly $\mathrm{im}(F^1 \mathbf{L}_{\mathrm{dR}})$ is locally the image of any generating section.

Before we can prove the proposition, we will need some finer information about the local structure of \mathcal{S}_K , which we will discuss below. The proof is provided right after (3.21).

3.17. We assume now that (L, Q) is perfect. For any \mathbb{F}_p -scheme S , let $(S/\mathbb{Z}_p)_{\mathrm{cris}}$ be the big crystalline site for S over $\mathrm{Spec} \mathbb{Z}_p$ (cf. [BM90, p. 178]), and let $\mathcal{O}_S^{\mathrm{cris}}$ be the structure sheaf of $(S/\mathbb{Z}_p)_{\mathrm{cris}}$. Recall that an object in $(S/\mathbb{Z}_p)_{\mathrm{cris}}$ is a triple (U, T, γ) , where U is an S -scheme, $U \hookrightarrow T$ is a nilpotent thickening of \mathbb{Z}_p -schemes with ideal of definition $\mathcal{J}_{(U \hookrightarrow T)}$, and γ is a divided power structure on $\mathcal{J}_{(U \hookrightarrow T)}$ that is compatible with the natural divided power structure on the ideal $p\mathcal{O}_T$. For any sheaf G over $(S/\mathbb{Z}_p)_{\mathrm{cris}}$, and any object (U, T, γ) in $(S/\mathbb{Z}_p)_{\mathrm{cris}}$, we denote by G_T the restriction of G to the fppf site over T .

Let $A_{\mathbb{F}_p}^{\text{KS}}$ be the fiber of A^{KS} over $\mathcal{S}_{K,\mathbb{F}_p}$. Let \mathbf{H}_{cris} be the first crystalline cohomology of $A_{\mathbb{F}_p}^{\text{KS}}$ over $\mathcal{S}_{K,\mathbb{F}_p}$. This is a crystal of locally free $\mathcal{O}_{\mathcal{S}_{K,\mathbb{F}_p}}^{\text{cris}}$ -modules over $(\mathcal{S}_{K,\mathbb{F}_p}/\mathbb{Z}_p)_{\text{cris}}$. We have a natural identification

$$\mathbf{H}_{\text{dR}} \otimes \mathbb{Z}_p = \varprojlim_n \mathbf{H}_{\text{cris}, \mathcal{S}_{K,\mathbb{Z}/p^n}}$$

of coherent sheaves over $\mathcal{S}_{K,\mathbb{Z}_p}$. In fact, for any \mathcal{S}_K -scheme T in which p is nilpotent, we have a canonical identification of coherent sheaves

$$\mathbf{H}_{\text{dR},T} = \mathbf{H}_{\text{cris},T}.$$

\mathbf{H}_{cris} has more structure: it is an F -crystal. More precisely, let Fr be the absolute Frobenius endomorphism on $\mathcal{S}_{K,\mathbb{F}_p}$. Then $\text{Fr}^* \mathbf{H}_{\text{cris}}$ is identified with the relative crystalline cohomology of the Frobenius pull-back $\text{Fr}^* A_{\mathbb{F}_p}^{\text{KS}}$, and the relative Frobenius map $A_{\mathbb{F}_p}^{\text{KS}} \rightarrow \text{Fr}^* A_{\mathbb{F}_p}^{\text{KS}}$ induces a map of crystals

$$\mathbf{F} : \text{Fr}^* \mathbf{H}_{\text{cris}} \rightarrow \mathbf{H}_{\text{cris}}.$$

If T is any $\mathcal{S}_{K,\mathbb{F}_p}$ -scheme, we get an induced map of coherent sheaves

$$\mathbf{F} : \text{Fr}^* \mathbf{H}_{\text{dR},T} \rightarrow \mathbf{H}_{\text{dR},T}.$$

The kernel of this map is precisely the Hodge filtration $\text{Fr}^* F^1 \mathbf{H}_{\text{dR},T} \subset \text{Fr}^* \mathbf{H}_{\text{dR},T}$.

3.18. Let $s \in \mathcal{S}_K(k)$ be a point valued in a perfect field k of characteristic p , let \widehat{U}_{x_0} be the completion of \mathcal{S}_K at s . Following [Kis10], we will now give an explicit description for \widehat{U}_s , as well as for the filtered F -crystal $\mathbf{H}_{\text{cris},\widehat{U}_s}$ over \widehat{U}_s .

Let $\mathcal{O} = W(k)$; then we the restriction \mathbf{H}_{cris} (resp. $\mathbf{H}_{\text{cris}}^{\otimes(1,1)}$) to $(\text{Spec } k(s)/\mathbb{Z}_p)_{\text{cris}}$ corresponds to the \mathcal{O} -module $H_{\text{cris}}^1(A_s^{\text{KS}}/\mathcal{O})$ (resp. $\text{End}(H_{\text{cris}}^1(A_s^{\text{KS}}/\mathcal{O}))$). If σ is the canonical Frobenius lift on \mathcal{O} , \mathbf{F} induces a map

$$\mathbf{F}_s : \sigma^* \mathbf{H}_{\text{cris},s} \rightarrow \mathbf{H}_{\text{cris},s},$$

giving $\mathbf{H}_{\text{cris},s}$ the structure of an F -crystal over \mathcal{O} . Conjugation by \mathbf{F}_s induces an F -isocrystal structure on $\mathbf{H}_{\text{cris},s} \left[\frac{1}{p} \right]^{\otimes(1,1)}$. Set $E_0 = \mathcal{O} \left[\frac{1}{p} \right]$, let E/E_0 be a finite totally ramified extension, and let \overline{E}/E be an algebraic closure of E .

Proposition 3.19. *Suppose that $\tilde{s} : \mathcal{O}_E \rightarrow \mathcal{S}_K$ is a lift of s , and let $\tilde{s}_{\overline{E}} \in \mathcal{S}_K(\overline{E})$ be the attached generic geometric point.*

(1) *The crystalline comparison isomorphism*

$$\mathbf{H}_{p,\tilde{s}_{\overline{E}}} \otimes_{\mathbb{Z}_p} B_{\text{cris}} \xrightarrow{\sim} \mathbf{H}_{\text{cris},s} \otimes_{\mathcal{O}} B_{\text{cris}}$$

respects grading, is C -equivariant and carries $\pi_{p,\tilde{s}_{\overline{E}}} \otimes 1$ to $\pi_{\text{cris},s} \otimes 1$, where $\pi_{\text{cris},s} \in \mathbf{H}_{\text{cris},s}^{\otimes(2,2)}$ is an F -invariant tensor that does not depend on the choice of lift \tilde{s} .

(2) *There exists a C -equivariant isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathcal{O} -modules*

$$H \otimes_{\mathbb{Z}(p)} \mathcal{O} \xrightarrow{\sim} \mathbf{H}_{\text{cris},s}$$

carrying π to $\pi_{\text{cris},s}$.

(3) *Set*

$$\mathbf{L}_{\text{cris},s} = \text{im } \pi_{\text{cris},s} \subset \mathbf{H}_{\text{cris},s}^{\otimes(1,1)}.$$

Then $\mathbf{L}_{\text{cris},s}$ is a perfect quadratic space over \mathcal{O} that is isometric to $L \otimes \mathcal{O}$.

- (4) We have $\mathbf{L}_{\mathrm{dR},s} = \mathbf{L}_{\mathrm{cris},s} \otimes k$, and the Hodge filtration $F^1 \mathbf{H}_{\mathrm{dR},s} \subset \mathbf{H}_{\mathrm{dR},s}$ is $\mathrm{GSpin}(\mathbf{L}_{\mathrm{dR},s})$ -split. More precisely, there exists a canonical isotropic line $F^1 \mathbf{L}_{\mathrm{dR},s} \subset \mathbf{L}_{\mathrm{dR},s}$ such that

$$F^1 \mathbf{H}_{\mathrm{dR},s} = \ker(F^1 \mathbf{L}_{\mathrm{dR},s}) = \mathrm{im}(F^1 \mathbf{L}_{\mathrm{dR},s}).$$

Proof. All this follows from results in [Kis10]. The main input is (1.6), which shows that $\mathrm{GSpin}(L)$ is the point-wise stabilizer in $\mathrm{GL}_C^+(H)$ of π . The comparison isomorphism is functorial; so it is C -equivariant and preserves $\mathbb{Z}/2\mathbb{Z}$ -gradings. The fact that $\pi_{\mathrm{cris},s}$ belongs to $\mathbf{H}_{\mathrm{cris},s}^{\otimes(1,1)}$ (and not just $\mathbf{H}_{\mathrm{cris},s}^{\otimes(1,1)} \left[\frac{1}{p} \right]$) follows from [Kis10, 1.3.6(1), 1.4.3(1)]. That $\pi_{\mathrm{cris},s}$ is determined independently of the lift \tilde{s} is shown during the course of the proof of [Kis10, 2.3.5] using a parallel transport argument.

Now, [Kis10, 1.4.3(3)] shows that there exists a C -equivariant graded isomorphism

$$\mathbf{H}_{p,\tilde{s}_{\overline{E}}} \otimes_{\mathbb{Z}_p} \mathcal{O} \xrightarrow{\sim} \mathbf{H}_{\mathrm{cris},s}$$

carrying $\pi_{p,\tilde{s}_{\overline{E}}}$ to $\pi_{\mathrm{cris},s}$. So to show (2) we only have to observe that there exists a C -equivariant graded isomorphism

$$H \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p \xrightarrow{\sim} \mathbf{H}_{p,\tilde{s}_{\overline{E}}}$$

carrying π to $\pi_{p,\tilde{s}_{\overline{E}}}$; cf. (3.14).

(3) is immediate from (2), and the first assertion of (4) follows from [Kis10, 1.4.3(4)]. The second assertion of (4) follows from the discussion in (1.13). \square

3.20. We can now describe the explicit models for \widehat{U}_s and $\mathbf{H}_{\mathrm{cris},\widehat{U}_s}$. Choose any co-character $\mu_0 : \mathbb{G}_m \otimes k \rightarrow \mathrm{GSpin}(\mathbf{L}_{\mathrm{dR},s})$ splitting the Hodge filtration, and let $\mu : \mathbb{G}_m \otimes \mathcal{O} \rightarrow \mathrm{GSpin}(\mathbf{L}_{\mathrm{cris},s})$ be any lift of μ_0 . It determines a lift $F^1 \mathbf{H}_{\mathrm{cris},s} \subset \mathbf{H}_{\mathrm{cris},s}$ of the Hodge filtration as well as a splitting

$$\mathbf{H}_{\mathrm{cris},s} = F^1 \mathbf{H}_{\mathrm{cris},s} \oplus \overline{F}^1 \mathbf{H}_{\mathrm{cris},s}.$$

Since μ factors through $\mathrm{GSpin}(\mathbf{L}_{\mathrm{cris},s})$, it induces a splitting

$$\mathbf{L}_{\mathrm{cris},s} = F^1 \mathbf{L}_{\mathrm{cris},s} \oplus \mathbf{L}_{\mathrm{cris},s}^0 \oplus \overline{F}^1 \mathbf{L}_{\mathrm{cris},s},$$

where $F^1 \mathbf{L}_{\mathrm{cris},s} \subset \mathbf{L}_{\mathrm{cris},s}$ is an isotropic line lifting $F^1 \mathbf{V}_{\mathrm{dR},s}$. We again have:

$$F^1 \mathbf{H}_{\mathrm{cris},s} = \ker(F^1 \mathbf{L}_{\mathrm{cris},s}) = \mathrm{im}(F^1 \mathbf{L}_{\mathrm{cris},s}).$$

As usual, we are viewing $\mathbf{L}_{\mathrm{cris},s}$ as a space of endomorphisms of $\mathbf{H}_{\mathrm{cris},s}$.

Let $U \subset \mathrm{GL}(\mathbf{H}_{\mathrm{cris},s})$ be the opposite unipotent attached to this splitting: Namely, it is the unipotent radical of the parabolic sub-group attached to the filtration $\overline{F}^1 \mathbf{H}_{\mathrm{cris},s}$. Its intersection U_G with $\mathrm{GSpin}(\mathbf{L}_{\mathrm{cris},s})$ is again the unipotent radical of a parabolic sub-group of $\mathrm{GSpin}(\mathbf{L}_{\mathrm{cris},s})$.

Let \widehat{U} (resp. \widehat{U}_G) be the completion of U (resp. U_G) along the identity section. Let R (resp. R_G) be the ring of formal functions on \widehat{U} (resp. \widehat{U}_G); then we have a surjection $R \twoheadrightarrow R_G$ of formally smooth \mathcal{O} -algebras.

Set $\mathbf{H}_R = \mathbf{H}_{\mathrm{cris},s} \otimes_{\mathcal{O}} R'$ and $\mathbf{H}_{R_G} = \mathbf{H}_{\mathrm{cris},s} \otimes_{\mathcal{O}} R_G$, and equip both with the constant filtrations arising from the filtration $F^1 \mathbf{H}_{\mathrm{cris},s}$. Choose compatible isomorphisms

$$R \xrightarrow{\sim} \mathcal{O}[[t_1, \dots, t_r]]; \quad R_G \xrightarrow{\sim} \mathcal{O}[[t_1, \dots, t_d]]$$

such that the identity sections are identified with the maps $t_i \mapsto 0$. Equip both R and R_G with Frobenius lifts $\varphi : t_i \mapsto t_i^p$. Equip \mathbf{H}_R with an F -crystal structure over R via:

$$F : \varphi^* \mathbf{H}_R = \sigma^* \mathbf{H}_{\mathrm{cris},s} \otimes_{\mathcal{O}} R \xrightarrow{F_s \otimes 1} \mathbf{H}_{\mathrm{cris},s} \otimes_{\mathcal{O}} R = \mathbf{H}_R \xrightarrow{g} \mathbf{H}_R,$$

where $g \in \widehat{U}(R)$ is the tautological element. This equips \mathbf{H}_{R_G} with the induced F -crystal structure under the map $R \rightarrow R_G$.

Let \widehat{U}' be the universal deformation space for the abelian variety A_s^{KS} ; then \widehat{U}_s is naturally identified with a closed formal sub-scheme of \widehat{U}' . The relative first crystalline cohomology of the universal formal abelian scheme over \widehat{U}' gives rise to a filtered F -crystal \mathbf{H}' over \widehat{U}' . The following is a more precise version of (3.8):

Theorem 3.21. *There exists a unique isomorphism of \widehat{U} with \widehat{U}' as formal \mathcal{O} -schemes under which the filtered F -crystal \mathbf{H}_R is identified with \mathbf{H}' . This induces an isomorphism between \widehat{U}_G and \widehat{U}_s .*

Proof. The first assertion is due to Faltings; cf. [Fal99, §7]. The second is shown during the course of the proof of [Kis10, 2.3.5]. \square

Proof of (3.16). All the assertions here are local on \mathcal{S}_K , and so we can work with the formal completion \widehat{U}_s at a point $s \in \mathcal{S}_K(\overline{\mathbb{F}}_p)$. By (3.21) above, we can identify \widehat{U}_s with \widehat{U}_s and $\mathbf{H}_{\text{dR}, \widehat{U}_s}$ with \mathbf{H}_{R_G} . The extension of $\pi_{\text{dR}, \mathbb{Q}}$ over \widehat{U}_s is now given by the constant section $\pi_{\text{cris}, s} \otimes 1$ of $\mathbf{H}_{R_G}^{\otimes(1,1)}$, thus showing (1). The remaining assertions are clear from the construction of \mathbf{H}_{R_G} . \square

3.22. Since $\mathcal{S}_{K, \mathbb{Z}_p}$ is smooth over \mathbb{Z}_p , \mathbf{H}_{cris} is in fact determined by $\mathbf{H}_{\text{dR}} \otimes \mathbb{Z}_p$ equipped with its Gauss-Manin connection. In particular, π_{dR} gives rise to a global section

$$\pi_{\text{cris}} \in H^0((\mathcal{S}_{K, \mathbb{F}_p}/\mathbb{Z}_p)_{\text{cris}}, \mathbf{H}_{\text{cris}}^{\otimes(2,2)}).$$

Again, we can view this as an idempotent endomorphism of the crystal $\mathbf{H}_{\text{cris}}^{\otimes(1,1)}$, and we denote by \mathbf{L}_{cris} the image of π_{cris} in $\mathbf{H}_{\text{cris}}^{\otimes(1,1)}$. If $s \rightarrow \mathcal{S}_{K, \mathbb{F}_p}$ is a point valued in a perfect field $k(s)$, then the restriction of \mathbf{L}_{cris} to $(\text{Spec } k(s)/\mathbb{Z}_p)_{\text{cris}}$ corresponds to the quadratic space $\mathbf{L}_{\text{cris}, s}$ seen above, and the restriction of π_{cris} corresponds to the tensor $\pi_{\text{cris}, s}$.

Let $\mathcal{O} = W(k(s))$, and suppose that we have a lift $\tilde{s} \in \mathcal{S}_K(\mathcal{O})$ of s . Then there is a natural isomorphism

$$\mathbf{H}_{\text{cris}, s} \xrightarrow{\sim} \mathbf{H}_{\text{dR}, \tilde{s}}.$$

This equips $\mathbf{H}_{\text{cris}, s}$ with a filtration $F^1 \mathbf{H}_{\text{cris}, s}$ that is **strongly divisible**. That is, we have:

$$\mathbf{F}_s \left(\sigma^* (p^{-1} F^1 \mathbf{H}_{\text{cris}, s} + \mathbf{H}_{\text{cris}, s}) \right) = \mathbf{H}_{\text{cris}, s}.$$

If we now endow $\mathbf{L}_{\text{cris}, s}$ with its induced filtration and $\mathbf{L}_{\text{cris}, s} \left[\frac{1}{p} \right]$ with the conjugation action of \mathbf{F}_s , then we again obtain a strongly divisible module, in the sense that the following identity holds:

$$(3.22.1) \quad \mathbf{F}_s \left(\sigma^* (p^{-1} F^1 \mathbf{L}_{\text{cris}, s} + F^0 \mathbf{L}_{\text{cris}, s} + p \mathbf{L}_{\text{cris}}) \right) = \mathbf{L}_{\text{cris}, s}.$$

Indeed, this follows from two facts: $\mathbf{H}_{\text{cris}, s}^{\otimes(1,1)}$ is strongly divisible [Laf80, 4.2]; and $\mathbf{L}_{\text{cris}, s} \subset \mathbf{H}_{\text{cris}, s}^{\otimes(1,1)}$ is the image of an \mathbf{F} -equivariant, filtration preserving projector $\pi_{\text{cris}, s}$.

In fact, this gives us more. If $R_G = \widehat{\mathcal{O}}_{\mathcal{S}_K, s}$ is as in (3.20), then $\mathbf{L}_{\text{dR}, R_G} = \text{im } \pi_{\text{dR}}|_{\mathbf{H}_{\text{dR}, R_G}^{\otimes(1,1)}}$ is also strongly divisible:

$$(3.22.2) \quad \mathbf{F} \left(\sigma^* (p^{-1} F^1 \mathbf{L}_{\text{dR}, R_G} + F^0 \mathbf{L}_{\text{cris}, s} + p \mathbf{L}_{\text{cris}}) \right) = \mathbf{L}_{\text{dR}, R_G}.$$

This follows from the explicit description of \mathbf{F} in terms of \mathbf{F}_s and the tautological element of $U_G(R_G)$. Note of course that \mathbf{F} is only defined on $\mathbf{L}_{\text{dR}, R_G} \left[\frac{1}{p} \right]$.

3.23. Over \mathcal{S}_K we now have two tautological line bundles: First, we have the Hodge or canonical bundle ω^{KS} attached to the top exterior power of the sheaf of differentials $\Omega_{A^{\text{KS}}/\mathcal{S}_K}^1$. Second, we have the line bundle $F^1 \mathbf{L}_{\text{dR}}$. These are closely related, as the next result shows.

We will need a little preparation. Fix an isotropic line $F^1 L \subset L$, and let $P \subset G$ be the parabolic sub-group stabilizing it. Let $F^1 H = \ker(F^1 L) \subset H$ be the corresponding isotropic sub-space of H . The G -torsor \mathcal{P}_{dR} introduced in (3.16)(2) has a natural reduction of structure group to P . Indeed, we can take $\mathcal{P}_{\text{dR},P}$ to be the sub-functor of \mathcal{P}_{dR} such that, for any \mathcal{S}_K -scheme T , we have:

$$\mathcal{P}_{\text{dR},P}(T) = \{\xi \in \mathcal{P}_{\text{dR}}(T) : \xi(F^1 H \otimes \mathcal{O}_T) = F^1 \mathbf{H}_{\text{dR},T}\}.$$

The proof of *loc. cit.* shows that this is indeed a P -torsor. Given such a torsor, one immediately gets a functor from $\mathbb{Z}_{(p)}$ -representations of the group scheme P to vector bundles over \mathcal{S}_K . More precisely, given a $\mathbb{Z}_{(p)}$ -representation U of P , we can view it as a trivial vector bundle over $\text{Spec } \mathbb{Z}_{(p)}$ with a P -action, and then take the corresponding \mathcal{S}_K -vector bundle to be the quotient $(\mathcal{P}_{\text{dR},T} \times_{\text{Spec } \mathbb{Z}_{(p)}} U)/P$, where P acts diagonally.

Proposition 3.24. *There exists a canonical isomorphism of line bundles:*

$$\omega^{\text{KS}, \otimes 2} \xrightarrow{\sim} ((F^1 \mathbf{L}_{\text{dR}})(-1))^{\otimes 2^{n+1}}.$$

Here, the (-1) denotes the twist by the (trivial) line bundle attached to the spinor character $\nu : P \rightarrow \mathbb{G}_m$. In particular, $F^1 \mathbf{L}_{\text{dR}}$ is a relatively ample line bundle for \mathcal{S}_K over $\mathbb{Z}_{(p)}$.

Proof. This follows from an argument of Maulik [Mau12, §5]. The main point is that both bundles involved are canonical extensions over the integral canonical model of automorphic line bundles.

ω^{KS} is the line bundle attached via $\mathcal{P}_{\text{dR},P}$ to the P -representation $\det(F^1 H)$, and $F^1 \mathbf{L}_{\text{dR}}(-1)$ is the line bundle attached to the representation $F^1 L(\nu)$.

The left multiplication map $L \otimes H \rightarrow H$ induces an isomorphism of P -representations

$$\text{gr}_F^{-1} L \otimes F^1 H \xrightarrow{\sim} \text{gr}_F^0 H.$$

Therefore, we have a canonical isomorphism of P -representations

$$\det(H) \xrightarrow{\sim} \det(F^1 H) \otimes \det(\text{gr}_F^0 H) \xrightarrow{\sim} \det(F^1 H)^{\otimes 2} \otimes (\text{gr}_F^{-1} L)^{\otimes 2^{n+1}}.$$

Since $(\text{gr}_F^{-1} L)^\vee \xrightarrow{\sim} F^1 L$, this gives us a canonical isomorphism of P -representations

$$\det(F^1 H)^{\otimes 2} \xrightarrow{\sim} (F^1 L)^{\otimes 2^{n+1}} \otimes \det(H).$$

On the other hand, the symplectic form ψ_δ on H induces a canonical isomorphism of P -representations

$$\text{gr}_0^F H \xrightarrow{\sim} (F^1 H)^\vee(\nu),$$

This shows that we have:

$$\det(H) \xrightarrow{\sim} \mathbb{Z}_{(p)}(\nu^{2^{n+1}}),$$

completing the proof of the claimed isomorphism.

The last statement of the lemma follows, since it is known that the bundle $\tilde{\omega}^{\text{KS}}$ is relatively ample; cf. for example [Lan08, 7.2.4.1(2)]. \square

4. SPECIAL ENDOMORPHISMS: GENERIC FIBER

All the notation is as in § 3.

4.1. Let T be an $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)_{\mathbb{C}}$ -scheme; then functoriality of cohomology gives us a natural map

$$\mathrm{End}(A_T^{\mathrm{KS}})_{(p)} \rightarrow H^0(T^{\mathrm{an}}, \mathbf{H}_B^{\otimes(1,1)}).$$

Definition 4.2. An endomorphism $f \in \mathrm{End}(A_T^{\mathrm{KS}})_{(p)}$ is **special** if it gives rise to a section of $\mathbf{L}_B \subset \mathbf{H}_B^{\otimes(1,1)}$ under the above map. It follows from the definition that f is special if and only if its fiber f_s at every point $s \rightarrow T$ is special. In fact, it is enough to require this for one point s in each connected component of T^{an} .

Let T be any $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ -scheme; then, for any prime ℓ , we have a natural map

$$\mathrm{End}(A_T^{\mathrm{KS}})_{(p)} \rightarrow H^0(T, \mathbf{H}_{\ell}^{\otimes(1,1)}).$$

Definition 4.3. Fix a prime $\ell \neq p$. An endomorphism $f \in \mathrm{End}(A_T^{\mathrm{KS}})_{(p)}$ is ℓ -**special** if it gives rise to a section of $\mathbf{V}_{\ell} \subset \mathbf{H}_{\ell}^{\otimes(1,1)}$ under the above map. We say that f is p -**special** if it gives rise to a section of $\mathbf{L}_p \subset \mathbf{H}_p^{\otimes(1,1)}$.

For any prime ℓ , we denote the space of ℓ -special endomorphisms by $L_{\ell}(A_T^{\mathrm{KS}})$.

One immediately sees that f is ℓ -special if and only if, in every connected component of T , there exists a point s such that the fiber f_s at s is ℓ -special. In particular, ℓ -specialness is a condition that can be checked at geometric points.

Lemma 4.4. *Suppose that T is an $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ -scheme and that $f \in \mathrm{End}(A_T^{\mathrm{KS}})_{(p)}$. Then the following are equivalent:*

- (1) f is ℓ -special for all primes ℓ .
- (2) f is ℓ -special for some prime ℓ .
- (3) The restriction of f over $T \otimes_{\mathbb{Q}} \mathbb{C}$ is special.

Proof. Let $s \rightarrow T$ be any \mathbb{C} -valued point. Then it is clear from the definitions that f_s is special if and only if it is ℓ -special for some (hence any) prime ℓ . Using this, the lemma easily follows. \square

Definition 4.5. Let T and f be as above. Then we say that f is **special** if it satisfies any of the equivalent conditions of (4.4). We denote the space of special endomorphisms of A_T^{KS} by $L(A_T^{\mathrm{KS}})$. Also set $V(A_T^{\mathrm{KS}}) = L(A_T^{\mathrm{KS}}) \left[\frac{1}{p} \right]$.

We have:

Proposition 4.6. *If $s \in \mathrm{Sh}_K(G_{\mathbb{Q}}, X)(\mathbb{C})$, then*

$$L(A_s^{\mathrm{KS}}) = \mathbf{L}_{B,s} \cap (\mathbf{L}_{B,s} \otimes \mathbb{C})^{(0,0)}.$$

In particular, for any $\mathrm{Sh}_K(G_{\mathbb{Q}}, X)$ -scheme T , $\mathrm{rk}_{\mathbb{Z}_{(p)}} L(A_T^{\mathrm{KS}}) \leq n$.

\square

4.7. Suppose that (\tilde{L}, \tilde{Q}) is another $\mathbb{Z}_{(p)}$ -quadratic space of signature $(n+d, 2)$ equipped with an embedding

$$(L, Q) \hookrightarrow (\tilde{L}, \tilde{Q}),$$

so that L is a direct summand of \tilde{L} . Let $\Lambda = L^{\perp} \subset \tilde{L}$; by assumption, it is positive definite over \mathbb{R} . Consider the Shimura datum $(\tilde{G}_{\mathbb{Q}}, \tilde{X})$, where \tilde{G} is the smooth $\mathbb{Z}_{(p)}$ -group scheme attached to \tilde{L} : We have an embedding

$$(G_{\mathbb{Q}}, X) \hookrightarrow (\tilde{G}_{\mathbb{Q}}, \tilde{X}).$$

Set $\tilde{K}_p = \tilde{G}(\mathbb{Z}_p)$, and let $\tilde{K}^p \subset \tilde{G}(\mathbb{A}_f^p)$ be a compact open with $K^p \subset \tilde{K}^p$. We then get a map

$$\mathrm{Sh}_K(G_{\mathbb{Q}}, X) \rightarrow \mathrm{Sh}_{\tilde{K}}(\tilde{G}_{\mathbb{Q}}, \tilde{X})$$

of Shimura varieties over \mathbb{Q} . Here, as usual, $\tilde{K} = \tilde{K}_p \tilde{K}^p$. For simplicity, we will write Sh_K for the first Shimura variety and $\tilde{\text{Sh}}_{\tilde{K}}$ for the second. We will also assume that \tilde{K}^p is small enough, so that $\tilde{\text{Sh}}_{\tilde{K}}$ is smooth, quasi-projective, and so that we have the Kuga-Satake abelian scheme $\tilde{A}_{\mathbb{Q}}^{\text{KS}}$ over it.

Since \tilde{C} is a projective module over C (1.2), the Serre tensor construction shows that the sheaf $\underline{\text{Hom}}_C(\tilde{C}, A_{\mathbb{Q}}^{\text{KS}})$ of C -equivariant maps from \tilde{C}^+ to A^{KS} (here, C acts on \tilde{C} via left translation) is represented by an abelian scheme over Sh_K .

Lemma 4.8. *$\underline{\text{Hom}}_C(\tilde{C}, A_{\mathbb{Q}}^{\text{KS}})$ has a natural $\mathbb{Z}/2\mathbb{Z}$ -grading, as well as a \tilde{C} action compatible with the grading. Moreover, there exists a canonical \tilde{C} -equivariant isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded abelian schemes over Sh_K :*

$$i : \tilde{A}_{\text{Sh}_K}^{\text{KS}} \xrightarrow{\sim} \underline{\text{Hom}}_C(\tilde{C}, A_{\mathbb{Q}}^{\text{KS}}).$$

Proof. The $\mathbb{Z}/2\mathbb{Z}$ -grading is simply the diagonal grading, and the action of \tilde{C} is via pre-composition by right multiplication.

The proof is now quite standard, and essentially comes down to the existence of a \tilde{C} -equivariant map of $\mathbb{Z}/2\mathbb{Z}$ -graded G -representations:

$$(4.8.1) \quad H \otimes_C \tilde{C} \rightarrow \tilde{H}$$

$$(4.8.2) \quad w \otimes z \mapsto w \cdot z.$$

Over $\text{Sh}_{K,\mathbb{C}}^{\text{an}}$, consider the variation of Hodge structures attached to the tuple

$$(\mathbf{H}_B \otimes_C \tilde{C}, \mathbf{H}_{\text{dR},\mathbb{C}}^{\text{an}} \otimes_C \tilde{C}, F^\bullet \mathbf{H}_{\text{dR},\mathbb{C}}^{\text{an}} \otimes_C \tilde{C}).$$

This is in fact the variation of Hodge structures attached to the abelian scheme $\underline{\text{Hom}}_C(\tilde{C}, A_{\text{Sh}_{K,\mathbb{C}}}^{\text{KS}})$ over $\text{Sh}_{K,\mathbb{C}}$.

The map (4.8.1) gives rise to an isomorphism of variations of Hodge structures:

$$(\mathbf{H}_B \otimes_C \tilde{C}, \mathbf{H}_{\text{dR},\mathbb{C}}^{\text{an}} \otimes_C \tilde{C}, F^\bullet \mathbf{H}_{\text{dR},\mathbb{C}}^{\text{an}} \otimes_C \tilde{C}) \xrightarrow{\sim} (\tilde{\mathbf{H}}_{B, \text{Sh}_{K,\mathbb{C}}^{\text{an}}}, \tilde{\mathbf{H}}_{\text{dR}, \text{Sh}_{K,\mathbb{C}}}^{\text{an}}, F^\bullet \tilde{\mathbf{H}}_{\text{dR}, \text{Sh}_{K,\mathbb{C}}}^{\text{an}});$$

and hence to an isomorphism of abelian schemes

$$i : \tilde{A}_{\text{Sh}_{K,\mathbb{C}}}^{\text{KS}} \xrightarrow{\sim} \underline{\text{Hom}}_C(\tilde{C}, A_{\text{Sh}_{K,\mathbb{C}}}^{\text{KS}}).$$

Consider the pro-variety:

$$\text{Sh}_{K^p} = \varprojlim_{H_p \subset K_p} \text{Sh}_{H_p K^p}(G, X).$$

This is a pro-Galois cover of Sh_K with Galois group K_p . For any connected component $S \subset \text{Sh}_K$, a geometric point $s \rightarrow S$ and a lift $\bar{s} \rightarrow \text{Sh}_{K_p}$, we obtain a map $j : \pi_1(S, s) \rightarrow K_p$, and the restriction of $\mathbf{H}_p \otimes_C \tilde{C} \xrightarrow{\sim} \tilde{\mathbf{H}}_{p, \text{Sh}_{K_p}}$ to S is the p -adic sheaf attached to the K_p -representation \tilde{H} , which we can view as a $\pi_1(S, s)$ -representation via j . Moreover, the map of p -adic cohomology groups induced by i is attached to the K_p -equivariant map (4.8.1), and is therefore also defined over S . From this, we conclude that i itself must be defined over Sh_K . \square

Proposition 4.9. *There is an isometric embedding*

$$\Lambda \hookrightarrow L(\tilde{A}_{\text{Sh}_K}^{\text{KS}})$$

such that there exist natural isometries:

- (1) $\mathbf{L}_B \xrightarrow{\sim} \Lambda^\perp \subset \tilde{\mathbf{L}}_{B, \text{Sh}_{K,\mathbb{C}}^{\text{an}}}$ of local systems over $\text{Sh}_{K,\mathbb{C}}^{\text{an}}$.
- (2) $\mathbf{L}_p \xrightarrow{\sim} \Lambda^\perp \subset \tilde{\mathbf{L}}_p$ of \mathbb{Z}_p -local systems over Sh_K .
- (3) $\mathbf{V}_{\mathbb{A}_f^p} \xrightarrow{\sim} \Lambda^\perp \subset \tilde{\mathbf{V}}_{\mathbb{A}_f^p, \text{Sh}_K}$ of \mathbb{A}_f^p -local systems over Sh_K .
- (4) $\mathbf{V}_{\text{dR}} \xrightarrow{\sim} \Lambda^\perp \subset \mathbf{V}_{\text{dR}, \text{Sh}_K}$ of filtered vector bundles with flat connection over Sh_K .

Moreover, for any Sh_K -scheme T , there exists a natural isometry of $\mathbb{Z}_{(p)}$ -quadratic spaces:

$$L(A_T^{\mathrm{KS}}) \xrightarrow{\sim} \Lambda^\perp \subset L(\tilde{A}_T^{\mathrm{KS}}).$$

Proof. Λ acts on $\tilde{A}_{\mathrm{Sh}_K}^{\mathrm{KS}} \xrightarrow{\sim} \underline{\mathrm{Hom}}_C(\tilde{C}, A_{\mathbb{Q}}^{\mathrm{KS}})$ as follows. We first consider the abelian scheme $\underline{\mathrm{Hom}}(\tilde{C}, A_{\mathbb{Q}}^{\mathrm{KS}})$: This has a \tilde{C} -action by left translations and contains $\tilde{A}_{\mathrm{Sh}_K}^{\mathrm{KS}}$ as a \tilde{C} -equivariant abelian sub-scheme by (4.8). Λ acts on this abelian scheme \tilde{C} -equivariantly via right translations. Unfortunately, since Λ does not quite commute with C within \tilde{C} , this naïve action does not preserve the sub-scheme $\tilde{A}_{\mathrm{Sh}_K}^{\mathrm{KS}}$. So we have to ‘twist’ the action of Λ a little.

The grading $\tilde{A}_{\mathrm{Sh}_K}^{\mathrm{KS},+} \times \tilde{A}_{\mathrm{Sh}_K}^{\mathrm{KS},-}$ allows us to decompose every section

$$f \in \tilde{A}_{\mathrm{Sh}_K}^{\mathrm{KS}} \subset \underline{\mathrm{Hom}}(\tilde{C}, A_{\mathbb{Q}}^{\mathrm{KS}})$$

as $f = f^+ + f^-$, where f^+ preserves gradings and f^- shifts them. For $v \in L$ and $z \in \tilde{C}$ we have $f(zv) = f(z)v$, which means that $f^\mp(zv) = f^\pm(z)v$. Now, given $\lambda \in \Lambda$, we set

$$(\lambda \cdot f)^+(z) = f^-(z\lambda); \quad (\lambda \cdot f)^-(z) = -f^+(z\lambda).$$

With this definition, we find, for $v \in L$,

$$(\lambda \cdot f)(zv) = f^-(zv\lambda) - f^+(zv\lambda) = f^+(z\lambda v) - f^-(z\lambda v) = (f^-(z\lambda) - f^+(z\lambda))v = (\lambda \cdot f)(z)v.$$

Here, we have used the identity $v\lambda + \lambda v = [v, \lambda]_{\tilde{C}} = 0$.

That this should be so is clear from the following: Consider the natural isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded \tilde{C} -modules

$$\alpha : H \otimes_C \tilde{C} \xrightarrow{\sim} \tilde{H},$$

Given $\lambda \in \Lambda$, $h \in H$ and $z \in \tilde{C}$, one checks:

$$\lambda h z = \begin{cases} \alpha(h \otimes \lambda z), & \text{if } h \in H^+; \\ \alpha(-h \otimes \lambda z), & \text{if } h \in H^-. \end{cases}$$

Now, the action of Λ gives us a Betti realization $\Lambda \rightarrow \mathbf{H}_B^{\otimes(1,1)}$. One checks easily that this is induced by a map of G -representations

$$\Lambda \rightarrow \tilde{L} \subset H^{\otimes(1,1)},$$

where G acts trivially on Λ , and $\tilde{L} \subset H^{\otimes(1,1)} = \mathrm{End}(H)$ is the inclusion induced from the action of \tilde{L} on H via left translation.

The enumerated statements about the various realizations of \mathbf{L} are now immediate.

Finally, given an Sh_K -scheme T , we obtain an embedding of $\mathbb{Z}_{(p)}$ -algebras:

$$\mathrm{End}_C(A_T^{\mathrm{KS}})_{(p)} \hookrightarrow \mathrm{End}_{\tilde{C}}(\tilde{A}_T^{\mathrm{KS}})_{(p)}.$$

This is defined as follows: Given an endomorphism f of A_T^{KS} , we obtain an endomorphism of $\tilde{A}_T^{\mathrm{KS}}$ carrying a map $\varphi : \tilde{C} \rightarrow A_T^{\mathrm{KS}}$ to the map $f \circ \varphi$. We have to show that this map induces an isomorphism

$$L(A_T^{\mathrm{KS}}) \xrightarrow{\sim} \Lambda^\perp \subset L(\tilde{A}_T^{\mathrm{KS}}).$$

But this can be checked on the level of cohomological realizations, where it is obvious. \square

5. p -SPECIAL ENDOMORPHISMS

We now assume that (L, Q) is perfect, and turn to the investigation of specialness over the model \mathcal{S}_K . For any \mathcal{S}_K -scheme T and any $\ell \neq p$, the definition of an ℓ -special endomorphism carries over directly from (4.3). We will now develop a version of p -specialness that works in general.

Definition 5.1. Suppose that $s \rightarrow \mathcal{S}_{K, \mathbb{F}_p}$ is a point valued in a perfect field $k(s)$. Then we obtain a map

$$\mathrm{End}(A_s^{\mathrm{KS}})_{(p)} \rightarrow \mathrm{End}(\mathbf{H}_{\mathrm{cris}, s}).$$

An endomorphism $f \in \mathrm{End}(A_s^{\mathrm{KS}})_{(p)}$ is **p -special** if it gives rise to an element of $\mathbf{L}_{\mathrm{cris}, s}$ under the above map.

Lemma 5.2. *Let T be a \mathcal{S}_K -scheme in which p is locally nilpotent, and suppose that we have $f \in \mathrm{End}(A_T^{\mathrm{KS}})_{(p)}$. Then the following are equivalent:*

- (1) *For every point $s \rightarrow T$ valued in a perfect field, the fiber $f_s \in \mathrm{End}(A_s^{\mathrm{KS}})_{(p)}$ is p -special.*
- (2) *In every connected component of T , there exists a point s valued in a perfect field such that the fiber f_s is p -special.*

Proof. This is an immediate consequence of the definition, the fact that the endomorphism scheme $\underline{\mathrm{End}}(A^{\mathrm{KS}})_{(p)}$ of A^{KS} is locally Noetherian, and (5.3) below, applied to the crystal $\mathbf{H}_{\mathrm{cris}}^{\otimes(1,1)}/\mathbf{L}_{\mathrm{cris}}$. \square

Lemma 5.3. *Suppose that T is a connected, locally Noetherian \mathbb{F}_p -scheme, and that F is a crystal of vector bundles over T equipped with a global section $e \in \Gamma((T/\mathbb{Z}_p)_{\mathrm{cris}}, F)$. Suppose that, for some point $x \rightarrow T$, the induced global section*

$$e_x \in \Gamma((\mathrm{Spec} k(x)/\mathbb{Z}_p)_{\mathrm{cris}}, F|_x)$$

vanishes. Then, for every point $y \rightarrow T$, the induced global section

$$e_y \in \Gamma((\mathrm{Spec} k(y)/\mathbb{Z}_p)_{\mathrm{cris}}, F|_y)$$

also vanishes.

Proof. Since T is connected, locally Noetherian, the lemma will follow if we can prove the following claim.

Claim. Suppose that x and y are points of T such that x is a specialization of y , and such that the prime ideal corresponding to y has height 1 in $\mathcal{O}_{T,x}$; then e_x vanishes if and only if e_y vanishes.

Let x^{perf} be the point attached to a perfect closure of $k(x)$. By [BM90, 1.3.5], e_x vanishes if and only if $e_{x^{\mathrm{perf}}}$ vanishes. So we can assume that $k = k(x)$ is perfect.

Let $\mathfrak{p}_y \subset \mathcal{O}_{T,x}$ be the prime ideal corresponding to y , and let R be the normalization of $\mathcal{O}_{T,x}/\mathfrak{p}_y$. By our hypotheses, R is an equicharacteristic complete DVR with residue field k , and so is isomorphic to $k[[t]]$. By pulling F back to $\mathrm{Spec} k[[t]]$, we are further reduced to the situation where $T = \mathrm{Spec} k[[t]]$, $x = \mathrm{Spec} k$ and $y = \mathrm{Spec} k((t))$. A crystal over T is now given by a finite free $W(k)[[t]]$ -module M equipped with a flat, topologically quasi-nilpotent connection $\nabla : M \rightarrow M \otimes \Omega_{W(k)[[t]]}^1$. In other words, we have a derivation $D : M \rightarrow M$ over $\frac{d}{dt}$ such that a sufficiently large iteration of D carries M into pM . The global sections of F are equal to the module of horizontal elements $M^{\nabla=0}$.

The corresponding crystal over x is the one attached to the $W(k)$ -module $M_0 = M/tM$. The restriction from global sections over T to global sections over x is just the reduction-mod- t map $M^{\nabla=0} \rightarrow M_0$. It is easy to see that this map is injective: Indeed, suppose that we have $m \in tM$

such that $D(m) = 0$, and suppose that $n \in \mathbb{Z}_{>0}$ is the largest integer such that $m \in t^n M$ (such an n exists if and only if m is non-zero). Write $m = t^n m'$, for some $m' \notin tM$. We then have:

$$0 = D(m) = D(t^n m') = nt^{n-1}m' + t^n D(m').$$

Dividing by t^{n-1} , this gives us $nm' \in tM$, which implies that $m' \in tM$, contradicting our assumption that m is non-zero. So we find that a global section of a crystal over T is 0 precisely if it restricts to 0 over x .

By [BM90, 1.3.5] again, restriction from global sections over T to global sections over y is injective. And so we find that a global section of a crystal over T is 0 precisely when it restricts to 0 over y . This proves the claim and the lemma. \square

Definition 5.4. Let T be an \mathcal{S}_K -scheme and let $f \in \text{End}(A_T^{\text{KS}})_{(p)}$. If $p\mathcal{O}_T = 0$, we will say that f is **p -special** if it satisfies the equivalent conditions of (5.2). In general, we will say that f is **p -special**, if its restriction to $T \otimes \mathbb{F}_p$ and $T \otimes \mathbb{Q}$ are both p -special.

We will say that f is **special** if it is ℓ -special for *every* prime ℓ .

Given any prime ℓ , write $L_\ell(A_T^{\text{KS}})$ for the space of ℓ -special endomorphisms, and $L(A_T^{\text{KS}})$ for the space of special endomorphisms. By definition, we have:

$$L(A_T^{\text{KS}}) = \bigcap_{\ell \text{ prime}} L_\ell(A_T^{\text{KS}}).$$

Remark 5.5. It is easy to see that any special endomorphism is fixed by the Rosati involution on $\text{End}(A_T^{\text{KS}})_{(p)}$ induced from λ^{KS} (check this for any of its ℓ -adic realizations). So the assignment $f \mapsto f \circ f$ defines a positive definite $\mathbb{Z}_{(p)}$ -quadratic form on $L(A_T^{\text{KS}})$.

We have:

Lemma 5.6. *Let T be an \mathcal{S}_K -scheme such that every generic point of $T \otimes \mathbb{F}_p$ is the specialization of a point in $T \otimes \mathbb{Q}$. Then an endomorphism $f \in \text{End}(A_T^{\text{KS}})_{(p)}$ is p -special over $T \otimes \mathbb{Q}$ if and only if it is p -special over $T \otimes \mathbb{F}_p$. In particular, in this situation, if f is p -special, then it is in fact special.*

Proof. First assume that $T = \text{Spec } \mathcal{O}_E$, for some complete discrete valuation ring \mathcal{O}_E with characteristic 0 fraction field E and characteristic p perfect residue field k . Let $s = \text{Spec } k$, and let $\overline{\eta} = \text{Spec } \overline{E}$, for some algebraic closure \overline{E}/E . Then the result holds because the p -adic comparison isomorphism for A_T^{KS} carries $L_{p,\overline{\eta}} \otimes B_{\text{cris}}$ into $L_{\text{cris},s} \otimes B_{\text{cris}}$ (cf. (3.19)(1)).

For general T , in every connected component of $T \otimes \mathbb{F}_p$, we can find a point s valued in an algebraically closed field that is the specialization of a point valued in a complete discrete valuation field. Now we can apply the result of the previous paragraph to s .

The last assertion follows because ℓ -specialness is independent of the prime ℓ in characteristic 0. \square

5.7. Set $k = k(x_0)$, $W = W(k)$, and let $\widehat{U} = \text{Spf } \widehat{\mathcal{O}}_{\mathcal{S}_K, x_0}$ be the completion of \mathcal{S}_K at x_0 ; let $\mathfrak{m} \subset \widehat{\mathcal{O}}_{\mathcal{S}_K, x_0}$ be its maximal ideal. Suppose that we have a *non-zero* p -special endomorphism $f_0 \in L_p(A_{x_0}^{\text{KS}}) \subset \text{End}(A_{x_0}^{\text{KS}})_{(p)}$, and consider the deformation functor

$$\begin{aligned} \text{Def}_{(x_0, f_0)} : \text{Art}_W &\rightarrow \text{Set} \\ \mathcal{O} &\mapsto \{(x, f) : x \in \widehat{U}(\mathcal{O}); f \in \text{End}(A_x^{\text{KS}})_{(p)} \text{ lifting } f_0\}. \end{aligned}$$

Here, Art_W is the category of local artinian W -algebras with residue field k .

5.8. Suppose that we have a surjection $\mathcal{O} \rightarrow \overline{\mathcal{O}}$ in Art_W , whose kernel I admits nilpotent divided powers. Suppose also that we have $(\overline{x}, \overline{f}) \in \text{Def}_{(x_0, f_0)}(\overline{\mathcal{O}})$ giving rise to an abelian scheme $A_{\overline{x}}^{\text{KS}}$ over $\overline{\mathcal{O}}$ equipped with a special endomorphism \overline{f} .

Let $\mathbf{H}_{\mathcal{O}}$ be the \mathcal{O} -module obtained by evaluating $\mathbf{H}_{\text{cris, Spec } \overline{\mathcal{O}}}$ on $\text{Spec } \mathcal{O}$, and let $\mathbf{L}_{\mathcal{O}} \subset \text{End}(\mathbf{H}_{\mathcal{O}})$ be the corresponding quadratic space. Denote by $\mathbf{H}_{\overline{\mathcal{O}}}$ and $\mathbf{L}_{\overline{\mathcal{O}}}$ the induced modules over $\overline{\mathcal{O}}$; then

$$\mathbf{H}_{\overline{\mathcal{O}}} = \mathbf{H}_{\text{dR}, \overline{x}}$$

is equipped with its Hodge filtration $F^1 \mathbf{H}_{\overline{\mathcal{O}}}$.

The endomorphism \overline{f} gives rise to an endomorphism of $\mathbf{H}_{\mathcal{O}}$, which we will denote $f_{\mathcal{O}}$: note that $f_{\mathcal{O}}$ lies in $\mathbf{L}_{\mathcal{O}}$. The reduction $f_{\overline{\mathcal{O}}} \in \mathbf{L}_{\overline{\mathcal{O}}}$ preserves the Hodge filtration $F^1 \mathbf{H}_{\overline{\mathcal{O}}}$ and so lies in $F^0 \mathbf{L}_{\overline{\mathcal{O}}}$.

By Serre-Tate (cf. [Kat81, 1.2.1]) and Grothendieck-Messing (cf. [Mes72, V.1.6]), we have a natural bijection

$$\left(\begin{array}{c} \text{Isomorphism classes of} \\ \text{abelian schemes over } \mathcal{O} \text{ lifting } A_{\overline{x}}^{\text{KS}} \end{array} \right) \xrightarrow{\sim} \left(\begin{array}{c} \text{Direct summands } F^1 \mathbf{H}_{\mathcal{O}} \subset \mathbf{H}_{\mathcal{O}} \\ \text{lifting } F^1 \mathbf{H}_{\overline{\mathcal{O}}} \end{array} \right)$$

This works as follows: For any lift A_x of $A_{\overline{x}}^{\text{KS}}$, we have an identification $H_{\text{dR}}^1(A_x/\mathcal{O}) = \mathbf{H}_{\mathcal{O}}$ that carries the Hodge filtration on $H_{\text{dR}}^1(A_x/\mathcal{O})$ to the corresponding summand $F_x^1 \mathbf{H}_{\mathcal{O}} \subset \mathbf{H}_{\mathcal{O}}$.

Proposition 5.9. *The bijection above induces further bijections*

$$\begin{aligned} \left(\text{Lifts } x \in \widehat{U}(\mathcal{O}) \text{ of } \overline{x} \right) &\xrightarrow{\sim} \left(\text{Isotropic lines } F^1 \mathbf{L}_{\mathcal{O}} \subset \mathbf{L}_{\mathcal{O}} \text{ lifting } F^1 \mathbf{L}_{\overline{\mathcal{O}}} \right); \\ \left(\text{Lifts } (x, f) \in \widehat{U}_{f_0}(\mathcal{O}) \text{ of } (\overline{x}, \overline{f}) \right) &\xrightarrow{\sim} \left(\begin{array}{c} \text{Isotropic lines } F^1 \mathbf{L}_{\mathcal{O}} \subset \mathbf{L}_{\mathcal{O}} \text{ lifting } F^1 \mathbf{L}_{\overline{\mathcal{O}}} \\ \text{and orthogonal to } f_{\mathcal{O}} \end{array} \right). \end{aligned}$$

Proof. In the first of the claimed bijections, there is a natural map in one direction: Given a lift $x \in \widehat{U}(\mathcal{O})$ and the identification $\mathbf{L}_{\text{dR}, x} = \mathbf{L}_{\mathcal{O}}$, the Hodge filtration $F^1 \mathbf{L}_{\text{dR}, x}$ gives us an isotropic line $F^1 \mathbf{L}_{\mathcal{O}}$ lifting $F^1 \mathbf{L}_{\overline{\mathcal{O}}}$. Further, \overline{f} lifts to an endomorphism of A_x^{KS} if and only if $f_{\mathcal{O}}$ preserves the Hodge filtration $F^1 \mathbf{H}_{\mathcal{O}}$. Since $F^1 \mathbf{H}_{\mathcal{O}}$ is the annihilator in $\mathbf{H}_{\mathcal{O}}$ of $F^1 \mathbf{L}_{\mathcal{O}}$, it is easy to see that $f_{\mathcal{O}}$ preserves $F^1 \mathbf{H}_{\mathcal{O}}$ if and only if it is orthogonal to $F^1 \mathbf{L}_{\mathcal{O}}$.

So it is enough to show that the first map is a bijection. For this, we can work successively with the thickenings $\mathcal{O}/I^{[r-1]} \rightarrow \mathcal{O}/I^{[r]}$ (where $I^{[r]}$ denotes the r^{th} -divided power of I), and assume that $I^2 = 0$. If $\mathfrak{m}_{\mathcal{O}} \subset \mathcal{O}$ is the maximal ideal, we can even work with successively with the thickenings $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{r-1}I \rightarrow \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r I$, and further assume that $\mathfrak{m}_{\mathcal{O}}I = 0$. In this case, we find that both sides of the map in question are vector spaces over k of the same dimension, namely $n \cdot \dim_k I$, and that the map is a map of k -vector spaces. Since it is clearly injective, we see that the map must in fact be a bijection. \square

Corollary 5.10. *Let the notation be as above. Suppose that we have a lift $x \in \widehat{U}(\mathcal{O})$ of \overline{x} corresponding to an isotropic line $F^1 \mathbf{L}_{\mathcal{O}} \subset \mathbf{L}_{\mathcal{O}}$. Let $J \subset \mathcal{O}$ be the smallest ideal such that \overline{f} lifts to an endomorphism of $A_x^{\text{KS}} \otimes_{\mathcal{O}} (\mathcal{O}/J)$. Then J is principal, and is generated by the element $[f_{\mathcal{O}}, w]$, for any basis element w of $F^1 \mathbf{L}_{\mathcal{O}}$.* \square

Set $R = \widehat{\mathcal{O}}_{\mathcal{K}, x_0}$; let $I_{f_0} \subset R$ be the ideal defining $\widehat{U}_{f_0} \subset \widehat{U}$, and set $R_{f_0} = R/I_{f_0}$, so that $\widehat{U}_{f_0} = \text{Spf } R_{f_0}$. The following result is essentially a retread of [Del81, Prop. 1.5].

Proposition 5.11. *$\text{Def}_{(x_0, f_0)}$ is pro-represented by a formal closed sub-scheme $\widehat{U}_{f_0} \subset \widehat{U}$ defined by a single equation $a_{f_0} \in \mathfrak{m}$. Moreover, $p \nmid a$, so that \widehat{U}_{f_0} is flat over \mathbb{Z}_p .*

Proof. Let $R' = R/\mathfrak{m}I_{f_0}$, so that $R' \rightarrow R_{f_0}$ is a surjection in ProArt_W with square-zero kernel I' . Note that I' admits trivial nilpotent divided powers. Let f be the universal lifting of f_0 over \widehat{U}_{f_0} . Then, by (5.10), $I' = I_{f_0}/\mathfrak{m}I_{f_0}$ is a principal ideal in R' . Nakayama's lemma now says that I_{f_0} is itself principal, generated by an element $a_{f_0} \in \mathfrak{m}$.

For the second statement, we use the argument from the proof of [Del81, 1.6]. As in *loc. cit.*, we reduce immediately to the following assertion: f_0 does not propagate to a special endomorphism of $\widetilde{A}_{R \otimes \mathbb{F}_p}^{\text{KS}}$. Suppose that such a propagation did exist; then we can consider its crystalline realization $f_R \in \mathbf{L}_{\text{dR}, R}$. Choose $k \in \mathbb{Z}_{\geq 0}$ minimal with respect to the condition that $p^{-k}f_R$ belongs to $\mathbf{L}_{\text{dR}, R}$. Now, $F(f_R) = f_R$, which implies that $F(p^{-k}f_R) = p^{-k}f_R$. By strong divisibility (3.22.1), $p^{-k}f_R$ lies in $F^0\mathbf{L}_{\text{dR}, R} + p\mathbf{L}_{\text{dR}, R}$. In particular, the image \bar{f}_R of $p^{-k}f_R$ in $\mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p}$ is a horizontal element that lies in $F^0\mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p}$. But (5.9) shows that the connection on $\mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p}$ induces an R -linear Kodaira-Spencer isomorphism

$$\text{gr}_F^0 \mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p} \xrightarrow{\sim} \text{gr}_F^{-1} \mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p} \otimes \widehat{\Omega}_{R \otimes \mathbb{F}_p/k}^1.$$

This shows that \bar{f}_R must actually lie in $F^1\mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p}$. But once again the connection on $\mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p}$ sets up an R -linear embedding (dual in a certain sense to the previous isomorphism):

$$F^1\mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p} \hookrightarrow \text{gr}_F^0 \mathbf{L}_{\text{dR}, R \otimes \mathbb{F}_p} \otimes \widehat{\Omega}_{R \otimes \mathbb{F}_p/k}^1.$$

This shows that $\bar{f}_R = 0$, which is a contradiction. \square

Corollary 5.12. *For every \mathcal{S}_K -scheme T and every prime ℓ , we have:*

$$L_p(A_T^{\text{KS}}) \hookrightarrow L_\ell(A_T^{\text{KS}}).$$

In particular, $L(A_T^{\text{KS}}) = L_p(A_T^{\text{KS}})$.

Proof. It follows from the definitions that it suffices to prove the corollary when T is a point $x_0 : \text{Spec } k \rightarrow \mathcal{S}_K$, with k a perfect field of characteristic p . If $f_0 \in L_p(A_{x_0}^{\text{KS}})$, then by (5.11), \widehat{U}_{f_0} is flat. This implies that there exists a finite extension $L/W_{\mathbb{Q}}$, and a lift $x : \text{Spec } \mathcal{O}_L \rightarrow \mathcal{S}_K$ of x_0 such that f_0 lifts to a special endomorphism f of A_x^{KS} . Now the argument in (5.6) shows that f_0 is ℓ -special for every ℓ . \square

Remark 5.13. From now on, we can and will refer to p -special endomorphisms simply as special endomorphisms.

Remark 5.14. One can deduce from results of Kisin towards the Langlands-Rapoport conjecture that we in fact have $L(A_T^{\text{KS}}) = L_\ell(A_T^{\text{KS}})$ for *any* prime ℓ ; cf. §7.

5.15. Let $\Lambda \subset L(A_{x_0}^{\text{KS}})$ be a $\mathbb{Z}_{(p)}$ -sub-module. We can consider the deformation functor $\text{Def}_{(x_0, \Lambda)}$, defined as follows: Fix a basis $\{f_{1,0}, \dots, f_{r,0}\}$ for Λ ; then, for any $B \in \text{Art}_W$, we have

$$\text{Def}_{(x_0, \Lambda)}(B) = \{(x, \{f_i : 1 \leq i \leq r\}) : x \in \widehat{U}(B); f_i \in \text{End}(A_x^{\text{KS}})_{(p)} \text{ lifting } f_{i,0}\}.$$

Clearly, this functor does not depend on the choice of basis for Λ . From (5.11), we see that $\text{Def}_{(x_0, \Lambda)}$ is pro-represented by a formal closed sub-scheme $\widehat{U}_\Lambda \subset \widehat{U}$ defined by r equations a_1, \dots, a_r , where, for each i , a_i is the equation for the deformation space for the special endomorphism f_i .

Proposition 5.16. *Suppose that the quadratic form restricted to $\Lambda^\perp \subset \mathbf{L}_{\text{cris}, x_0}$ is not divisible by p . Then the formal closed sub-scheme \widehat{U}_Λ has at least one irreducible component that is flat over W .*

Proof. We can assume that k is separably closed. Let $\mathcal{O}_L = W[\sqrt{p}]$: we claim that we can find an isotropic line $F^1(\Lambda^\perp \otimes_W \mathcal{O}_L)$ lifting $F^1\mathbf{L}_{\text{dR}, x_0}$. Granting this, the proof of the proposition is now simple: Since the kernel of the map $\mathcal{O}_L \rightarrow k$ has divided powers, we can identify

$\mathbf{L}_{\text{cris},x_0} \otimes_W \mathcal{O}_L$ with $\mathbf{L}_{\text{cris},\mathcal{O}_L}$. If $p > 3$, then the divided powers are topologically nilpotent and we can use the line $F^1(\Lambda^\perp \otimes \mathcal{O}_L)$ and (5.9) to produce a lift $x \in \widehat{U}_\Lambda(\mathcal{O}_L)$. Even if $p = 3$, we can first lift from k to $\mathcal{O}_L/(p)$, and then to \mathcal{O}_L . This shows that $\widehat{U}_\Lambda(\mathcal{O}_L)$ is non-empty, and so \widehat{U}_Λ has at least one flat component.

It remains to prove the claim, which is a simple exercise. In appropriate co-ordinates, the quadratic form on Λ^\perp can be written as

$$X_1^2 + X_2^2 + \cdots + X_r^2 + a_1 Y_1^2 + \cdots + a_d Y_d^2,$$

for $a_i \in pW$ and $r \geq 1$. There are two possibilities: First, the isotropic line $F^1 \mathbf{L}_{\text{dR},x_0}$ can lie in the plane given by the equations $X_j = 0$, for $j = 1, \dots, r$. Suppose its homogeneous co-ordinates are $[0 : 0 : \cdots : 0 : b_1 : \cdots : b_d]$; then we can choose arbitrary lifts $\tilde{b}_i \in W$ of the co-ordinates b_i , and take our lift over \mathcal{O}_L to be the one with co-ordinates $[x : 0 : \cdots : 0 : \tilde{b}_1 : \cdots : \tilde{b}_d]$, where

$$x^2 = - \sum_i a_i \tilde{b}_i^2.$$

The other possibility is that one of the co-ordinates X_i does not vanish at $F^1 \mathbf{L}_{\text{dR},x_0}$. In this case, $F^1 \mathbf{L}_{\text{dR},x_0}$ lies in the smooth locus of the quadric of isotropic lines in Λ^\perp , and so can be even lifted over W . \square

6. REGULAR INTEGRAL MODELS

We will revert to the notation from (4.7). We will assume in addition that (\tilde{L}, \tilde{Q}) is a perfect quadratic space. The next result shows that such a space always exists.

Lemma 6.1. *There exists a perfect quadratic lattice (\tilde{L}, \tilde{Q}) over $\mathbb{Z}_{(p)}$, and an embedding*

$$(L, Q) \hookrightarrow (\tilde{L}, \tilde{Q})$$

of quadratic lattices carrying L onto a direct summand of \tilde{L} such that $\Lambda = L^\perp$ is positive definite.

Proof. For any $a \in \mathbb{Z}_{(p)}$, let $\langle a \rangle$ be the rank 1-quadratic $\mathbb{Z}_{(p)}$ -module $\mathbb{Z}_{(p)}$ equipped with the quadratic form with value a on 1.

Let $v \in L$ be such that $\text{ord}_p(Q(v))$ is minimal. Then one easily checks that $\text{ord}_p([w_1, w_2]_Q) \geq \text{ord}_p(Q(v))$, for all $w_1, w_2 \in L$. In particular, for any $w \in L$, the projection $w - \frac{2[w,v]_Q}{[v,v]_Q}v$ onto $\langle v \rangle^\perp \subset L$ is well-defined, and we obtain an orthogonal decomposition

$$L = \langle v \rangle \oplus \langle v \rangle^\perp.$$

Applying this iteratively, we find that (L, Q) can be diagonalized, so that it is isometric to a lattice of the form

$$\bigoplus_{i=1}^n \langle a_i \rangle \bigoplus_{j=1}^2 \langle -b_j \rangle,$$

where a_i and b_j positive integers for varying i and j .

The proof is now quite simple, once we admit the classical fact (due to Lagrange) that every positive integer can be expressed as a sum of four squares. This implies that every positive integer can be expressed *primitively* as the sum of five squares. Indeed, given such an integer n , express $n - 1$ as the sum of four squares, and n as the sum of those squares and 1. Thus we see that $\langle a \rangle$ is a direct summand of $E_+ = \langle 1 \rangle^{\oplus 5}$, whenever $a > 0$. An easy argument then shows that $\langle a \rangle$ is a direct summand of $E_- = \langle 1 \rangle^{\oplus 5} \oplus \langle -1 \rangle$, whenever $a < 0$.

We now find that L embeds isometrically as a direct summand of $\tilde{L} = E_+^{\oplus n} \oplus E_-^{\oplus 2}$. \square

6.2. Let $\widetilde{\mathcal{S}}_{\widetilde{K}}$ be the integral canonical model for $\widetilde{\text{Sh}}_{\widetilde{K}}$ over $\mathbb{Z}_{(p)}$ (cf. 3.7). Assume that K and \widetilde{K} are small enough. Then, over $\widetilde{\mathcal{S}}_{\widetilde{K}}$, we have the Kuga-Satake abelian scheme $(\widetilde{A}^{\text{KS}}, \widetilde{\lambda}^{\text{KS}}, [\widetilde{\eta}^{\text{KS}}])$. Recall that we have a map of algebras

$$\widetilde{\rho} : \widetilde{C} \rightarrow \text{End}(\widetilde{A}_{\mathbb{Q}}^{\text{KS}})_{(p)}.$$

Here, \widetilde{C} is the Clifford algebra of \widetilde{V} .

Just as in Section 3, we have the cohomology sheaf $\widetilde{\mathbf{H}}_{\mathbb{A}_f^p}$ over $\widetilde{\mathcal{S}}_{\widetilde{K}}$. Let $\widetilde{\pi} \in \widetilde{H}^{\otimes(2,2)}$ be the idempotent projector with image $\widetilde{V} \subset \widetilde{H}^{\otimes(1,1)}$. Then we get the corresponding global section

$$\widetilde{\pi}_{\mathbb{A}_f^p} \in H^0(\widetilde{\mathcal{S}}_{\widetilde{K}}, \widetilde{\mathbf{H}}_{\mathbb{A}_f^p}^{\otimes(2,2)}).$$

As in (3.11), we can define étale sheaves

$$I_{\widetilde{G}}^p \subset I^p(\widetilde{A}^{\text{KS}}, \widetilde{\lambda}^{\text{KS}}).$$

Here, $I_{\widetilde{G}}^p$ is the étale sheaf of polarization preserving graded isomorphisms

$$\eta : \widetilde{H} \otimes \mathbb{A}_f^p \xrightarrow{\sim} \mathbf{H}_{\mathbb{A}_f^p}$$

that are \widetilde{C} -equivariant and carry $\widetilde{\pi}$ to $\widetilde{\pi}_{\mathbb{A}_f^p}$. Recall from (3.12) that $[\widetilde{\eta}^{\text{KS}}]$ is the image of a \widetilde{K}^p -level structure $[\widetilde{\eta}_{\widetilde{G}}^{\text{KS}}]$, a section of $I_{\widetilde{G}}^p/\widetilde{K}^p$.

6.3. Fix a $\mathbb{Z}_{(p)}$ -basis $\underline{v} = \{v_1, \dots, v_d\}$ for Λ . For any $\widetilde{\mathcal{S}}_{\widetilde{K}}$ -scheme T , a \underline{v} -**structure** for T is a collection of special endomorphisms $\underline{f} = \{f_1, \dots, f_d\} \subset V(\widetilde{A}_T^{\text{KS}})$ such that $[f_i, f_j] = [v_i, v_j]_{\widetilde{Q}}$, for all pairs i, j . Given an \underline{v} -structure \underline{f} on T , we let $I_{\underline{f}}^p \subset I_{\widetilde{G}}^p|_T$ be the sub-sheaf of isomorphisms η that carry v_i to f_i , for $i = 1, \dots, d$. Then $I_{\underline{f}}^p$ is naturally a torsor under the locally constant sheaf of groups $\underline{G}_{\mathbb{A}_f^p}$. A K^p -**level structure** on (T, \underline{f}) is a section $[\eta_{\underline{f}}^p]$ of $I_{\underline{f}}^p/K^p$ over T mapping to $[\widetilde{\eta}_{\widetilde{G}}^p]$ under the obvious map

$$I_{\underline{f}}^p/K^p \rightarrow (I_{\widetilde{G}}^p/\widetilde{K}^p)_T.$$

Consider the functor on $\widetilde{\mathcal{S}}_{\widetilde{K}}$ -schemes:

$$T \mapsto \left\{ (\underline{f}, [\eta_{\underline{f}}^p]) : \underline{f} \text{ a } \underline{v}\text{-structure for } T; [\eta_{\underline{f}}^p] \text{ a } K^p\text{-level structure for } (T, \underline{f}) \right\}.$$

Proposition 6.4. *The above functor is represented by a finite and unramified $\widetilde{\mathcal{S}}_{\widetilde{K}}$ -scheme $Z_{K^p}(\Lambda)$.*

Proof. This is standard. The only thing to note is that the locus where an endomorphism of $\widetilde{A}^{\text{KS}}$ is special is open and closed. \square

Proposition 6.5.

- (1) Sh_K is equipped with a canonical \underline{v} -structure $\underline{f} = \{f_1, \dots, f_d\} \subset L(\widetilde{A}_{\text{Sh}_K}^{\text{KS}})$ and a canonical K^p -level structure $[\widetilde{\eta}_{\underline{f}}^{\text{KS}}]$.
- (2) The induced map $\text{Sh}_K \rightarrow Z_{K^p}(\Lambda)$ identifies Sh_K with a union of connected components of $Z_{K^p}(\Lambda)_{\mathbb{Q}}$.

Proof. (4.9) gives us a canonical copy $\Lambda \subset V(\widetilde{A}_{\text{Sh}_K}^{\text{KS}})$, and the chosen basis \underline{v} for Λ gives us a canonical \underline{v} -structure \underline{f} over Sh_K .

Let $\underline{f}_{\mathbb{A}_f^p}$ be the étale realization of the \underline{v} -structure \underline{f} . By construction, we have an orthogonal decomposition of \mathbb{A}_f^p -sheaves over Sh_K :

$$\widetilde{\mathbf{V}}_{\mathbb{A}_f^p} = \mathbf{V}_{\mathbb{A}_f^p} \perp \langle \underline{f}_{\mathbb{A}_f^p} \rangle.$$

We also have a \tilde{C} -equivariant isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{A}_f^p -sheaves over Sh_K :

$$(6.5.1) \quad H_{\mathbb{A}_f^p} \otimes_C \tilde{C} \xrightarrow{\sim} \widetilde{H}_{\mathbb{A}_f^p}$$

$$(6.5.2) \quad v_i \otimes w \mapsto \mathbf{f}_{i, \mathbb{A}_f^p}(w).$$

From this, we find that giving a K^p -level structure $[\tilde{\eta}_{\underline{f}}^{\mathrm{KS}}]$ is equivalent to giving the K^p -level structure $[\eta_G^{\mathrm{KS}}]$ as in (3.13).

We now move on to the second assertion. More generally, let $(\underline{v}', g) \in \tilde{L}^d \times \tilde{G}(\mathbb{A}_f^p)$ be a pair such that $g(\underline{v}) = \underline{v}'$. Attached to such a pair is the smooth sub-group $G' \subset \tilde{G}$ that stabilizes \underline{v}' , and the associated Shimura variety $\mathrm{Sh}_{K'}(G'_{\mathbb{Q}}, X')$, where $K' = g\tilde{K}g^{-1} \cap G'(\mathbb{A}_f^p)$. Just as above, one sees that there exists a natural map

$$\mathrm{Sh}_{K'}(G'_{\mathbb{Q}}, X') \rightarrow Z_{K^p}(\Lambda),$$

whose image depends only on the class of (\underline{v}', g) in the space

$$\tilde{G}(\mathbb{Z}_{(p)}) \backslash \tilde{L}^d \times \tilde{G}(\mathbb{A}_f^p) / \tilde{K}^p.$$

One can now easily check (by base-changing to \mathbb{C} , for example) that $Z_{K^p}(\Lambda)_{\mathbb{Q}}$ is the union of the images of the varieties $\mathrm{Sh}_{K'}(G'_{\mathbb{Q}}, X')$ for varying pairs (\underline{v}', g) , and that each of these images is an open and closed sub-variety. \square

6.6. Over $Z_{K^p}(\Lambda)$, we have the tautological \underline{v} -structure \underline{f} . For $i = 1, \dots, d$, let $\mathbf{f}_{i, \mathrm{dR}}$ be the de Rham realization of the special endomorphism \mathbf{f}_i : It is a global section of \mathbf{L}_{dR} over $Z_{K^p}(\Lambda)$. Let $\mathbf{\Lambda}_{\mathrm{dR}} \subset \tilde{\mathbf{L}}_{\mathrm{dR}, Z_{K^p}(\Lambda)}$ be the coherent sub-sheaf generated by the $\mathbf{f}_{i, \mathrm{dR}}$: it is a trivial vector bundle over $Z_{K^p}(\Lambda)$ of rank d .

Lemma 6.7.

- (1) *There exists an open sub-scheme $Z_{K^p}^{\mathrm{pr}}(\Lambda) \subset Z_{K^p}(\Lambda)$ such that $T \rightarrow Z_{K^p}(\Lambda)$ factors through $Z_{K^p}^{\mathrm{pr}}(\Lambda)$ if and only if the pull-back of $\tilde{\mathbf{L}}_{\mathrm{dR}, Z_{K^p}(\Lambda)} / \mathbf{\Lambda}_{\mathrm{dR}}$ to T is a vector bundle of rank n . We have:*

$$Z_{K^p}^{\mathrm{pr}}(\Lambda)_{\mathbb{Q}} = Z_{K^p}(\Lambda)_{\mathbb{Q}}.$$

- (2) *Let k be a perfect field of characteristic p and let $W = W(k)$, and suppose that we have $\tilde{s} : \mathrm{Spec} W \rightarrow Z_{K^p}(\Lambda)$ such that the restriction to $\mathrm{Spec} W_{\mathbb{Q}}$ factors through Sh_K . Then \tilde{s} factors through $Z_{K^p}^{\mathrm{pr}}(\Lambda)$.*
- (3) *Suppose that T is smooth over $\mathbb{Z}_{(p)}$ and that $f : T \rightarrow Z_{K^p}(\Lambda)$ is such that $f|_{T_{\mathbb{Q}}}$ factors through Sh_K . Then f factors through $Z_{K^p}^{\mathrm{pr}}(\Lambda)$.*
- (4) *If L is maximal, then $Z_{K^p}^{\mathrm{pr}}(\Lambda) = Z_{K^p}(\Lambda)$.*

Proof. We begin with (1): The inclusion $\mathbf{\Lambda}_{\mathrm{dR}} \subset \tilde{\mathbf{L}}_{\mathrm{dR}}$ induces a map

$$\mathcal{O}_{Z_{K^p}(\Lambda)} \xrightarrow{\sim} \wedge^d \mathbf{\Lambda}_{\mathrm{dR}} \rightarrow \wedge^d \tilde{\mathbf{L}}_{\mathrm{dR}},$$

and hence a section $e \in H^0(Z_{K^p}(\Lambda), \wedge^d \tilde{\mathbf{L}}_{\mathrm{dR}})$. One easily checks now that $Z_{K^p}^{\mathrm{pr}}(\Lambda)$ is the locus where this section does not vanish.

Next, we consider (2). Choose an algebraic closure $\overline{E}/W(k)_{\mathbb{Q}}$. By (6.9) below, it suffices to show that the étale realization $\mathbf{\Lambda}_{p, \tilde{s}_{\overline{E}}} \subset \tilde{\mathbf{L}}_{p, \tilde{s}_{\overline{E}}}$ is a direct summand. But, $\mathbf{\Lambda}_p$ is globally a direct summand of $\tilde{\mathbf{L}}_{p, \tilde{s}_{\overline{E}}}$, since its inclusion in the latter is induced by the map of K_p -representations $\Lambda \hookrightarrow \tilde{L}$.

(3) now follows: Indeed, (1) and (2) show that $U := f^{-1}(Z_{K^p}^{\mathrm{pr}}(\Lambda))$ is an open sub-scheme of T containing $T_{\mathbb{Q}}$, and through which all the $W(\overline{\mathbb{F}}_p)$ -valued points of T factor. Since T is smooth over $\mathbb{Z}_{(p)}$, this implies that all the $\overline{\mathbb{F}}_p$ -points of T factor through U , and so U must be all of T .

Now assume that L is maximal. Choose any point $s \rightarrow Z_{K^p}(\Lambda)$ valued in a finite field. We will show that the fibers $\{\mathbf{f}_{i,\mathrm{dR},s}\}$ of the de Rham realizations generate a vector sub-space of $\tilde{\mathbf{L}}_{\mathrm{dR},s}$ of rank d . To do this, we will consider the crystalline realizations $f_i = \mathbf{f}_{i,\mathrm{cris},s} \in \tilde{\mathbf{L}}_{\mathrm{cris},s}$, and show that they generate a direct summand.

Using (3.19)(3), we can identify $\tilde{\mathbf{L}}_{\mathrm{cris},s}$ isometrically with $\tilde{\mathbf{L}}_{\mathcal{O}}$, where $\mathcal{O} = W(k)$. The space $\Lambda_{\mathrm{dR},\tilde{s}} \subset \tilde{\mathbf{L}}_{\mathrm{dR},\tilde{s}}$ is by hypothesis isometric to $\Lambda_{\mathcal{O}}$. So it follows from (2.3)(3) that there are two possibilities: Either it is a direct summand of $\tilde{\mathbf{L}}_{\mathcal{O}}$, in which case we are done. Or $\Lambda_{\tilde{f}}^{\perp} \subset \tilde{\mathbf{L}}_{\mathcal{O}}$ is a perfect lattice. In this latter situation, one finds from the argument in (5.16) that there exists a lift $\tilde{s} : \mathrm{Spec} W \rightarrow Z_K(\Lambda)$ of s . Now, we can apply the argument from (2), since, by the maximality of $\Lambda_{\mathbb{Z}_p}$, the étale realization $\Lambda_p \subset \tilde{\mathbf{L}}_p$ has to be a direct summand over all of $Z_K(\Lambda)_{\mathbb{Q}}$. \square

Lemma 6.8. *Assume k is algebraically closed, and let M be a finite free module over W such that $M \left[\frac{1}{p} \right]$ is equipped with a direct summand $M^0 \subset M$ and an isomorphism*

$$F : \sigma^* M \left[\frac{1}{p} \right] \xrightarrow{\sim} M \left[\frac{1}{p} \right]$$

such that $F(\sigma^ M^0) \subset M$. Then the image of the natural map*

$$(M^0)^{F=1} \otimes_{\mathbb{Z}_p} W \rightarrow M$$

is saturated in M .

Proof. First, assume that $M^0 = M$; then F induces a map $F : \sigma^* M \rightarrow M$.

It follows from the Dieudonné-Manin classification of F -crystals over W [Man62] that there exists a largest F -stable direct summand $M^{\mathrm{ét}} \subset M$ such that F induces an isomorphism $\sigma^* M^{\mathrm{ét}} \xrightarrow{\sim} M^{\mathrm{ét}}$. Clearly, $(M^0)^{F=1} \otimes W$ maps into $M^{\mathrm{ét}}$. So, replacing M with $M^{\mathrm{ét}}$, we can assume that F is an isomorphism (M is then called a **unit root crystal**). In this situation, it is known [Kat73, 4.1.1] that the map

$$(M^0)^{F=1} \otimes_{\mathbb{Z}_p} W \rightarrow M$$

is an isomorphism.

Note that all we needed to apply the Dieudonné-Manin classification was for F to be injective.

For general M , set

$$M' = \{m \in M^0 : F^i(m) \in M^0, \text{ for all } i \in \mathbb{Z}_{\geq 0}\}.$$

We need to explain what we mean by $F^i(m)$. One defines this inductively: We set $F(m) = F(\sigma^* m)$ and $F^i(m) = F(\sigma^* F^{i-1}(m))$, where we are using the assumption that $F^{i-1}(m) \in M^0$ in each inductive step.

Now, F restricts to a (necessarily injective) map $\sigma^* M' \rightarrow M'$. Moreover, M' is a direct summand of M , and $(M^0)^{F=1} \otimes W$ clearly maps into M' . So, replacing M by M' , we are reduced to the situation considered above. \square

Corollary 6.9. *Suppose that k is a perfect field of characteristic $p > 0$ (we allow $p = 2$). Let G be a p -divisible group over $W = W(k)$, and let $D(G)$ be the Dieudonné module for G over W . It is a filtered F -crystal. Then the image of the natural map*

$$\mathrm{End}(G) \otimes_{\mathbb{Z}_p} W \rightarrow \mathrm{End}_W(D(G))$$

is saturated. In particular, if A is an abelian scheme over W then $\mathrm{End}(A_{W_{\mathbb{Q}}}) \otimes W$ embeds in $\mathrm{End}_W(H_{\mathrm{dR}}^1(A/W))$ as a direct summand.

Proof. Since the endomorphisms of G defined over k form a direct summand of the group of endomorphisms of $G_{k'}$ for any finite extension k'/k , we can assume that k is in fact algebraically closed.

Set $M = \text{End}_W(D(G))$: We have a direct summand $M^0 \subset M$ consisting of endomorphisms that preserve the Hodge filtration $\text{Fil}^1 D(G)$, and the conjugation action of the semi-linear Frobenius on $D(G)$ induces an isomorphism $F : \sigma^* M \left[\frac{1}{p} \right] \xrightarrow{\sim} M \left[\frac{1}{p} \right]$ such that $F(\sigma^* M^0) \subset M$. The last condition follows from the strong divisibility of the F -crystal $D(G)$ (cf. 3.22). It is well-known that we have a canonical isomorphism of \mathbb{Z}_p -modules:

$$\text{End}(G) \xrightarrow{\sim} (M^0)^{F=1}.$$

So the result follows from (6.8).

For the last assertion, let $G = A[p^\infty]$ be the associated p -divisible group. Since $H_{\text{dR}}^1(A/W)$ can be canonically identified with $\mathbb{D}(G)(W)$, we only have to observe that, by the Nerónian property, $\text{End}(A_{W_{\mathbb{Q}}}) = \text{End}(A)$, and that $\text{End}(A) \otimes \mathbb{Z}_p$ is a direct summand of $\text{End}(G)$, since any element of $\text{End}(A)$ that acts trivially on the p -torsion $A[p]$ has to be divisible by p . \square

6.10. Suppose that we are given a point $x_0 \in Z_{K^p}^{\text{pr}}(\Lambda)(k)$ valued in a perfect field k of characteristic $p > 0$. Let \underline{f}_0 be the attached \underline{v} -structure on $x_0 : \text{Spec } k \rightarrow \mathcal{S}_{\tilde{K}}$, and let $\Lambda \subset L(\tilde{A}_{x_0}^{\text{KS}})$ be the sub-space spanned by \underline{f} . Let $h_0 \in \Lambda^\perp \subset L(\tilde{A}_{x_0}^{\text{KS}})$ be another special endomorphism. Then the intersection of the deformation space $\text{Def}_{(x_0, h_0)}$ with $(Z_{K^p}(\Lambda))_{x_0}^\wedge$ within $(\mathcal{S}_{\tilde{K}})_{x_0}^\wedge$ can be identified with $\text{Def}_{(x_0, \Lambda + \langle h_0 \rangle)}$.

Proposition 6.11. *Suppose that the quadratic form restricted to $(\Lambda + \langle h_0 \rangle)^\perp \subset L_{\text{cris}, x_0}$ is not divisible by p . Then at least one irreducible component of $\text{Def}_{(x_0, \Lambda + \langle h_0 \rangle)}$ is flat over \mathbb{Z}_p .*

Proof. Follows from (5.16). \square

Remark 6.12. Note that $(\Lambda + \langle h_0 \rangle)^\perp \subset \Lambda^\perp$ is a rank $n + 1$ direct summand. If the quadratic form were divisible by p on this sub-space, then $\Lambda_{\mathbb{F}_p}^\perp$ would contain an isotropic sub-space of codimension 1. Since $\text{rad}(\Lambda_{\mathbb{F}_p}^\perp)$ has the same dimension as $\text{rad}(L_{\mathbb{F}_p})$, the maximal isotropic sub-space of $\Lambda_{\mathbb{F}_p}^\perp$ has dimension

$$\left\lfloor \frac{n+2}{2} \right\rfloor + \dim \text{rad}(L_{\mathbb{F}_p}).$$

So, if this quantity is smaller than $n + 1$, the hypothesis of (6.11) will always hold.

6.13. Recall from (3.16) that we have a canonical \tilde{G} -torsor $\tilde{\mathcal{P}}_{\text{dR}}$ over $\mathcal{S}_{\tilde{K}}$ consisting of \tilde{G} -structure preserving trivializations of $\tilde{\mathbf{H}}_{\text{dR}}$. Let $(\underline{f}, [\eta_{\underline{f}}^{\text{KS}}])$ be the tautological \underline{v} -structure over $Z(\Lambda)$, and let $\mathcal{P}_{\text{dR}, \Lambda} \subset \tilde{\mathcal{P}}_{\text{dR}, Z_{K^p}(\Lambda)}$ be the sub-functor such that, for any $Z_{K^p}(\Lambda)$ -scheme T , we have:

$$\mathcal{P}_{\text{dR}, \Lambda}(T) = \{\xi \in \tilde{\mathcal{P}}_{\text{dR}}(T) : \xi \circ v_i = \underline{f}_{i, \text{dR}} \circ \xi, \text{ for } i = 1, \dots, d\}.$$

Proposition 6.14.

(1) *The restriction of $\mathcal{P}_{\text{dR}, \Lambda}$ over $Z_{K^p}^{\text{pr}}(\Lambda)$ is a G -torsor.*

(2) *The map $p_2 : \mathcal{P}_{\text{dR}, \Lambda} \rightarrow M_G^{\text{loc}}$, given, for any $\mathbb{Z}_{(p)}$ -scheme T , by:*

$$\begin{aligned} \mathcal{P}_{\text{dR}, \Lambda}(T) &\rightarrow M_G^{\text{loc}}(T) \\ (x, \xi) &\mapsto \xi^{-1}(F^1 \tilde{\mathbf{L}}_{\text{dR}}). \end{aligned}$$

is smooth of relative dimension $\dim G_{\mathbb{Q}}$. Here, $(x, \xi) \in \mathcal{P}_{\text{dR}, \Lambda}(T)$ lies over a point $x \in Z_{K^p}(\Lambda)(T)$.

Proof. (1) is an easy consequence of (2.2)(4). (2) is immediate from (5.9). \square

Remark 6.15. The proposition gives us a *local model diagram* in the terminology of [RZ96, DP94, Pap00].

Corollary 6.16. $Z_K^{\text{pr}}(\Lambda)$ is lci and flat of relative dimension n over $\mathbb{Z}_{(p)}$. Moreover, $Z_K^{\text{pr}}(\Lambda)_{\mathbb{F}_p}$ is reduced if and only if $n \geq r$. It is normal if and only if $n \geq r + 1$, and is smooth if and only if $r = 0$. In particular, if $n \geq r$, then $Z_K^{\text{pr}}(\Lambda)$ is normal.

Proof. Follows from (2.7) and (6.14). \square

Corollary 6.17. Suppose that L is maximal; then $Z_{K^p}(\Lambda)$ is a healthy regular $\mathbb{Z}_{(p)}$ -scheme.

Proof. This is a consequence of (6.14) and (2.7)(4). \square

Definition 6.18. A pro-scheme X over $\mathbb{Z}_{(p)}$ satisfies the **extension property** if, for any healthy regular $\mathbb{Z}_{(p)}$ -scheme S , any map $S \otimes \mathbb{Q} \rightarrow X$ extends to a map $S \rightarrow X$.

Definition 6.19. A model \mathcal{S}_{K^p} for Sh_{K^p} over $\mathbb{Z}_{(p)}$ is an **integral canonical model** for Sh_{K^p} if it is healthy regular and has the extension property. If \mathcal{S}_{K^p} is an integral canonical model for Sh_{K^p} , and if $K = K_p K^p$ is a compact open in $G(\mathbb{A}_f)$, then we will call $\mathcal{S}_K := \mathcal{S}_{K^p}/K^p$ the **integral canonical model** for Sh_K .

Theorem 6.20. Suppose that L is maximal. Then the pro-Shimura variety Sh_{K^p} admits an integral canonical model \mathcal{S}_{K^p} over $\mathbb{Z}_{(p)}$.

Proof. Choose any perfect quadratic space \tilde{L} over $\mathbb{Z}_{(p)}$ containing L as a direct summand, and such that $\Lambda = L^\perp \subset \tilde{L}$ is positive definite. This is always possible by (6.1). Let \mathcal{S}_{K^p} be the Zariski closure of Sh_{K^p} in

$$\varprojlim_{K^p \subset G(\mathbb{A}_f^p)} Z_{K^p}^{\text{pr}}(\Lambda).$$

By (6.5) and (6.17), \mathcal{S}_{K^p} is healthy regular. It also satisfies the extension property: Indeed, we already know that $\tilde{\mathcal{S}}_{\tilde{K}^p}$ has the extension property by [Kis10, 2.3.8]. So this result follows from the extension property for endomorphisms for abelian schemes over regular bases; cf. [FC90, I.2.7]. \square

Remark 6.21. In particular, \mathcal{S}_{K^p} is uniquely determined by Sh_{K^p} . It depends only on L and not on the choice of perfect quadratic space \tilde{L} containing L .

Proposition 6.22. Choose a small enough compact open $K^p \subset G(\mathbb{A}_f^p)$, and set $K = K_p K^p$. The geometric special fiber $\mathcal{S}_{K, \mathbb{F}_p}$ is an lci scheme over \mathbb{F}_p of dimension n . Let r be the dimension of the radical of the quadratic space $L_{\mathbb{F}_p}$. Then:

- (1) $\mathcal{S}_{K, \mathbb{F}_p}$ is reduced if and only if $n \geq r$.
- (2) $\mathcal{S}_{K, \mathbb{F}_p}$ is normal if and only if $n \geq r + 1$.
- (3) $\mathcal{S}_{K, \mathbb{F}_p}$ is smooth if and only if $r = 0$.

Proof. Follows from the construction and (6.16). \square

6.23. We will now revert to a general lattice L , but we will assume that $n \geq r$. By (6.16), $Z_K^{\text{pr}}(\Lambda)$ is a normal scheme. Let $\mathcal{S}_K^{\text{pr}}$ be the Zariski closure of Sh_K in $Z_K^{\text{pr}}(\Lambda)$: by (6.5), it is an open and closed sub-scheme of the latter, and is therefore a normal scheme over $\mathbb{Z}_{(p)}$. The formula:

$$\mathbf{V}_{\mathbb{A}_f^p} = \langle \underline{f}_{\mathbb{A}_f^p} \rangle^\perp \subset \tilde{\mathbf{V}}_{\mathbb{A}_f^p}$$

gives the unique extension of $\mathbf{V}_{\mathbb{A}_f^p, \text{Sh}_K}$ over $\mathcal{S}_K^{\text{pr}}$.

We can also extend $(V_{\mathrm{dR}}, F^\bullet V_{\mathrm{dR}})$ over $\mathcal{S}_K^{\mathrm{pr}}$ in two different ways. We set

$$L_{\mathrm{dR}} = \Lambda_{\mathrm{dR}}^\perp \subset \tilde{L}_{\mathrm{dR}}.$$

We also have:

$$L_{\mathrm{dR}}^\vee = \frac{\tilde{L}_{\mathrm{dR}}}{\Lambda_{\mathrm{dR}}}.$$

There is a symmetric, horizontal pairing Q_{dR} on L_{dR} arising from the restriction of the pairing on \tilde{L}_{dR} . It is of course no longer perfect, but there is instead a natural perfect pairing between L_{dR} and L_{dR}^\vee .

6.24. We also have crystalline realizations L_{cris} and L_{cris}^\vee over $\mathcal{S}_{K, \mathbb{F}_p}^{\mathrm{pr}}$. Set

$$\begin{aligned} L_{\mathrm{cris}} &= \Lambda_{\mathrm{cris}}^\perp \subset \tilde{L}_{\mathrm{cris}}; \\ L_{\mathrm{cris}}^\vee &= \frac{\tilde{L}_{\mathrm{cris}}}{\Lambda_{\mathrm{cris}}}. \end{aligned}$$

By construction, there is a perfect pairing of crystals between L_{cris} and L_{cris}^\vee .

Given a point $s \rightarrow \mathcal{S}_{K, \mathbb{F}_p}^{\mathrm{pr}}$ valued in a perfect field $k(s)$, with $\mathcal{O} = W(k(s))$, we obtain \mathcal{O} -modules $L_{\mathrm{cris}, s}$ and $L_{\mathrm{cris}, s}^\vee$. We have an identification of F -isocrystals $L_{\mathrm{cris}, s} \left[\frac{1}{p} \right] = L_{\mathrm{cris}, s}^\vee \left[\frac{1}{p} \right]$ over $E_0 = \mathcal{O} \left[\frac{1}{p} \right]$ and a perfect pairing

$$L_{\mathrm{cris}, s} \otimes L_{\mathrm{cris}, s}^\vee \rightarrow \mathcal{O}$$

that is F -invariant over L_0 . Suppose that E/E_0 is a totally ramified finite extension and that $\tilde{s} : \mathcal{O}_E \rightarrow Z_K^{\mathrm{pr}}(\Lambda)$ is a lift of s .

Proposition 6.25. *Let \overline{E}/E be an algebraic closure and let $\tilde{s}_{\overline{E}}$ be the geometric generic fiber of \tilde{s} . Then there are canonical comparison isomorphisms compatible with pairings:*

$$\begin{aligned} L_{p, \tilde{s}_{\overline{E}}} \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} &\xrightarrow{\sim} L_{\mathrm{cris}, s} \otimes_{\mathcal{O}} B_{\mathrm{cris}}; \\ L_{p, \tilde{s}_{\overline{E}}}^\vee \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} &\xrightarrow{\sim} L_{\mathrm{cris}, s}^\vee \otimes_{\mathcal{O}} B_{\mathrm{cris}}. \end{aligned}$$

Proof. This is immediate from the definition of the sheaves, (3.19)(1), and the functoriality of the comparison isomorphism. \square

6.26. Recall that we had the Kuga-Satake family $(A_{\mathbb{Q}}^{\mathrm{KS}}, \lambda_{\mathbb{Q}}^{\mathrm{KS}}, [\eta^{\mathrm{KS}}])$ over Sh_K , attached to the symplectic representation $\tilde{H}_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$.

Proposition 6.27. *Given any healthy regular $\mathbb{Z}_{(p)}$ -scheme T equipped with an étale map $T \rightarrow \mathcal{S}_K^{\mathrm{pr}}$, the restriction of the Kuga-Satake family $(A_{\mathbb{Q}}^{\mathrm{KS}}, \lambda_{\mathbb{Q}}^{\mathrm{KS}}, [\eta^{\mathrm{KS}}])$ to T extends uniquely to a family $(A_T^{\mathrm{KS}}, \lambda_T^{\mathrm{KS}}, [\eta^{\mathrm{KS}}])$ over T . In particular, if L is maximal then the Kuga-Satake family has a unique extension over the integral canonical model \mathcal{S}_K .*

Proof. Since T is healthy regular, it is sufficient to extend $(A_{T_{\mathbb{Q}}}^{\mathrm{KS}}, \lambda_{T_{\mathbb{Q}}}^{\mathrm{KS}})$ over co-dimension 1 points. This is done using the usual Néron-Ogg-Shafarevich criterion for good reduction. In fact, let $\mathbf{H}_{\mathbb{A}_f^p, \mathrm{Sh}_K}^p$ be the relative first étale cohomology sheaf of $A_{\mathbb{Q}}^{\mathrm{KS}}$ over Sh_K with coefficients in \mathbb{A}_f^p . Then we only have to note that the isomorphism $\mathbf{H}_{\mathbb{A}_f^p, \mathrm{Sh}_K}^p \otimes_C \tilde{C} \xrightarrow{\sim} \tilde{\mathbf{H}}_{\mathbb{A}_f^p, \mathrm{Sh}_K}^p$ allows us to view $\mathbf{H}_{\mathbb{A}_f^p, \mathrm{Sh}_K}^p$ as a sub-sheaf of $\tilde{\mathbf{H}}_{\mathbb{A}_f^p, \mathrm{Sh}_K}^p$, which has trivial inertial action at every co-dimension 1 point of $\mathcal{S}_K^{\mathrm{pr}}$. \square

6.28. Fix a healthy regular scheme T with an étale map $T \rightarrow \mathcal{S}_K^{\text{pr}}$ as above. Let $\mathbf{H}_{\mathbb{A}_f^p, T}$ be the first relative étale cohomology of A_T^{KS} over T with coefficients in \mathbb{A}_f^p . Then the \tilde{C} -equivariant isomorphism in (6.5.1) makes sense over T , and we can identify $\mathbf{V}_{\mathbb{A}_f^p, T}$ with a sub-sheaf of $\mathbf{H}_{\mathbb{A}_f^p, T}^{\otimes(1,1)}$.

Let $\mathbf{H}_{\text{cris}, T}$ be the relative first crystalline cohomology sheaf of $A_{T_{\mathbb{F}_p}}^{\text{KS}}$ over $T_{\mathbb{F}_p}$.

Proposition 6.29. *The natural embedding of vector bundles*

$$\mathbf{V}_{\text{dR}, T_{\mathbb{Q}}} \subset \mathbf{H}_{\text{dR}, T_{\mathbb{Q}}}^{\otimes(1,1)}$$

induces an embedding of vector bundles over T :

$$\mathbf{L}_{\text{dR}, T} \subset \mathbf{H}_{\text{dR}, T}^{\otimes(1,1)}.$$

This underlies an embedding of crystals of vector bundles over $T_{\mathbb{F}_p}$:

$$\mathbf{L}_{\text{cris}, T} \subset \mathbf{H}_{\text{cris}, T}^{\otimes(1,1)}.$$

Proof. Since T is regular, by [FC90, I.2.7], the isomorphism i from (4.8) extends uniquely:

$$i_T : \tilde{A}_T^{\text{KS}} \xrightarrow{\sim} \underline{\text{Hom}}_C(\tilde{C}, A_T^{\text{KS}})$$

over T . This induces an identification of crystals over $(T_{\mathbb{F}_p}/\mathbb{Z}_p)_{\text{cris}}$ with \tilde{C} -action:

$$\mathbf{H}_{\text{cris}, T} \otimes_C \tilde{C} \xrightarrow{\sim} \tilde{\mathbf{H}}_{\text{cris}, T}.$$

In turn this gives an embedding of crystals (End is the internal End in the category of crystals over $(T_{\mathbb{F}_p}/\mathbb{Z}_p)_{\text{cris}}$):

$$\begin{aligned} \underline{\text{End}}_C(\mathbf{H}_{\text{cris}, T}) &\rightarrow \underline{\text{End}}_{\tilde{C}}(\tilde{\mathbf{H}}_{\text{cris}, T}) \\ f &\mapsto (h \otimes \tilde{z} \mapsto f(h)\tilde{z}). \end{aligned}$$

Under this embedding, the sub-crystal $\underline{\text{End}}_C^+(\mathbf{H}_{\text{cris}, T})$ of grading-preserving endomorphisms maps onto the \tilde{C} -equivariant endomorphisms of $\tilde{\mathbf{H}}_{\text{cris}, T}$ that commute with Λ_{cris} , and the sub-crystal $\underline{\text{End}}_C^-(\mathbf{H}_{\text{cris}, T})$ of grading-shifting endomorphisms maps onto the \tilde{C} -equivariant endomorphisms of $\tilde{\mathbf{H}}_{\text{cris}}$ that anti-commute with $\Lambda_{\text{cris}, T}$.

Therefore, we obtain an embedding

$$\mathbf{L}_{\text{cris}, T} \subset \underline{\text{End}}_C^-(\mathbf{H}_{\text{cris}, T}) \subset \mathbf{H}_{\text{cris}}^{\otimes(1,1)}.$$

□

6.30. Given (6.29), the notions of ℓ -specialness and p -specialness for an endomorphism of A_T^{KS} carry over verbatim from Section 4. Denote the space of ℓ -special endomorphisms of A_T^{KS} by $L_{\ell}(A_T^{\text{KS}})$. Let $L(A_T^{\text{KS}})$ be the space of special endomorphisms, where ‘special’ means ℓ -special for every ℓ .

Proposition 6.31. *There exists a canonical isometry*

$$L(A_T^{\text{KS}}) \xrightarrow{\sim} \Lambda^{\perp} \subset L(\tilde{A}_T^{\text{KS}})$$

compatible with all cohomological realizations.

Proof. Just as in the proof of (4.9), the isomorphism i_T in (6.29) gives us an embedding

$$L_{\ell}(A_T^{\text{KS}}) \hookrightarrow \text{End}(\tilde{A}_T^{\text{KS}})_{(p)}.$$

Checking on the level of cohomological realizations (for $\ell \neq p$) shows that this induces an isometry:

$$L_{\ell}(A_T^{\text{KS}}) \xrightarrow{\sim} \Lambda^{\perp} \subset L_{\ell}(\tilde{A}_T^{\text{KS}}).$$

Moreover, for each closed point $s \rightarrow \mathcal{S}_K^{\text{pf}}$, we have an isometry of $W(k(s))_{\mathbb{Q}}$ -modules:

$$\mathbf{L}_{\text{cris},s} \xrightarrow{\sim} \Lambda^{\perp} \subset \tilde{\mathbf{L}}_{\text{cris},s}.$$

This shows that $L_p(A_T^{\text{KS}})$ maps onto $\Lambda^{\perp} \subset L_p(\tilde{A}_T^{\text{KS}}) = L(\tilde{A}_T^{\text{KS}})$. Putting all this together, we obtain the isometry claimed in the proposition. \square

Corollary 6.32. *Suppose that*

$$\left\lfloor \frac{n+2}{2} \right\rfloor + r \leq n.$$

Then, for any point $x_0 \rightarrow T$ valued in a perfect field $k(x_0)$ of characteristic p and any non-zero special endomorphism $f_0 \in L(A_{x_0}^{\text{KS}})$, the deformation functor $\text{Def}_{(x_0, f_0)}$ ⁴ is represented by a closed formal sub-scheme of \hat{T}_{x_0} , at least one component of which is flat over \mathbb{Z}_p .

Proof. This follows from (6.11) and (6.12). \square

7. ℓ -INDEPENDENCE

We maintain the notation from § 6, and we assume in addition that L is a maximal lattice, so that for suitable level $K^p \subset G(\mathbb{A}_f^p)$, we have the associated integral canonical model \mathcal{S}_K over $\mathbb{Z}_{(p)}$.

7.1. Suppose that we have $s \in \mathcal{S}_K(\overline{\mathbb{F}}_p)$, and suppose that s in fact arises from a point s_0 defined over the finite field \mathbb{F}_{p^r} . Then, for each $\ell \neq p$ and each m such that $r|m$, the p^m -power Frobenius Fr_m acts on $\mathbf{H}_{\ell,s}$ and on $\mathbf{V}_{\ell,s}$.

Set $\mathbb{Z}_p^{\text{nr}} = W(\overline{\mathbb{F}}_p)$ and let \mathbb{Q}_p^{nr} be its fraction field. We have the crystalline realization $\mathbf{L}_{\text{cris},s}$: this is a \mathbb{Z}_p^{nr} -module of endomorphisms of the \mathbb{Z}_p^{nr} -module $\mathbf{H}_{\text{cris},s}$. Set $\mathbf{V}_{\text{cris},s} = \mathbf{L}_{\text{cris},s} \left[\frac{1}{p} \right]$, and let $\mathbf{V}_{\text{cris},s}^{\mathbf{F}_s=1} \subset \mathbf{V}_{\text{cris},s}$ denote the \mathbb{Q}_p -subspace of \mathbf{F}_s -equivariant endomorphisms. For any prime ℓ , set

$$r_{\ell} = \begin{cases} \dim_{\mathbb{Q}_{\ell}}(\varinjlim_{r|m} \mathbf{V}_{\ell,s}^{\text{Fr}_m=1}), & \text{if } \ell \neq p; \\ \dim_{\mathbb{Q}_p} \mathbf{V}_{\text{cris},s}^{\mathbf{F}_s=1}, & \text{if } \ell = p. \end{cases}$$

From now on, we will maintain:

Assumption 7.2 (ℓ -independence). r_{ℓ} is independent of ℓ .

Remark 7.3. This assumption should always hold by results of Kisin [Kis]. Also, by [KM74], it will hold if one can realize $\{\{\mathbf{V}_{\ell,s}\}_{\ell \neq p}, \mathbf{V}_{\text{cris},s}\}$ as the family of cohomological realizations of a motive over $\overline{\mathbb{F}}_p$.

Theorem 7.4.

(1) *If $\ell \neq p$, the natural map*

$$L(A_s^{\text{KS}}) \otimes \mathbb{Q}_{\ell} \rightarrow \varinjlim_{r|m} \mathbf{V}_{\ell,s}^{\text{Fr}_m=1}$$

is an isometry of \mathbb{Q}_{ℓ} -quadratic spaces.

(2) *The natural map*

$$L(A_s^{\text{KS}}) \otimes \mathbb{Q}_p \rightarrow \mathbf{L}_{\text{cris},s}^{\mathbf{F}_s=1} \otimes \mathbb{Q}_p$$

is an isometry of \mathbb{Q}_p -quadratic spaces.

In particular, for any prime ℓ , $L_{\ell}(A_s^{\text{KS}}) = L(A_s^{\text{KS}})$.

⁴This is defined just as in Section 5.

Remark 7.5. Given our standing assumption (7.2), the numbered assertions of the theorem are equivalent to the following statement: $\mathrm{rk} L(A_s^{\mathrm{KS}}) = r$, where $r = r_\ell$, for one (hence any) prime ℓ .

The proof of this theorem will be given below following (7.11). First, we make a reduction.

Lemma 7.6. *We can assume that L is perfect and that $L(A_s^{\mathrm{KS}}) \neq 0$.*

Proof. Choose any non-zero positive definite quadratic space Λ over $\mathbb{Z}_{(p)}$ such that $\tilde{V} = V \oplus \Lambda_{\mathbb{Q}}$ admits a perfect $\mathbb{Z}_{(p)}$ -lattice \tilde{L} . Attached to this is a map of integral canonical models $\mathcal{S}_K \rightarrow \tilde{\mathcal{S}}_{\tilde{K}}$ (cf. the construction in (6.20)). Let \tilde{s} be the image of s in $\tilde{\mathcal{S}}_{\tilde{K}}$. Set

$$\tilde{r}_\ell = \begin{cases} \dim_{\mathbb{Q}_\ell}(\varinjlim_{r|m} \tilde{V}_{\ell, \tilde{s}}^{\mathrm{Fr}_m=1}), & \text{if } \ell \neq p; \\ \dim_{\mathbb{Q}_p} \tilde{V}_{\mathrm{cris}, s}^{\mathrm{Fr}_s=1}, & \text{if } \ell = p. \end{cases}$$

Then, by construction (6.23, 6.24), we have, for all ℓ , $\tilde{r}_\ell = r_\ell + \mathrm{rk} \Lambda$. Therefore, the assumption (7.2) holds for $s \in \mathcal{S}_K(\overline{\mathbb{F}}_p)$ if and only if it holds for $\tilde{s} \in \tilde{\mathcal{S}}_{\tilde{K}}(\overline{\mathbb{F}}_p)$.

Moreover, by (6.31), we have

$$L(A_s^{\mathrm{KS}}) = \Lambda^\perp \subset L(\tilde{A}_s^{\mathrm{KS}}).$$

So we find that (7.4) holds for \tilde{s} if and only if it holds for s . □

7.7. For $\ell \neq p$ and all $m \in \mathbb{Z}_{>0}$ such that $r \mid m$, let $I_{\ell, m} \subset \mathrm{GSpin}(\mathbf{V}_{\ell, s})$ be the commutant of Fr_m . Since Fr_m is a semi-simple element, $I_{\ell, m}$ is a reductive sub-group of $\mathrm{GSpin}(\mathbf{V}_{\ell, s})$. In fact, for m large enough $I_{\ell, m}$ does not depend on m , and we will denote it simply by I_ℓ . From now on, we will fix m such that $I_{\ell, m} = I_\ell$. Note that, for such an m , and any $m' \geq m$ with $r \mid m'$, we have

$$\mathbf{V}_{\ell, s}^{\mathrm{Fr}_{m'}=1} = \mathbf{V}_{\ell, s}^{\mathrm{Fr}_m=1}.$$

Lemma 7.8. *For every $\ell \neq p$, $\mathbf{V}_{\ell, s}^{\mathrm{Fr}_m=1}$ is an absolutely irreducible representation of I_ℓ .*

Proof. Let $q = p^m$. Fix $\ell \neq p$, and let $1, \alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1} \in \overline{\mathbb{Q}_\ell}$ be the distinct eigenvalues of Fr_m acting on $\mathbf{V}_{\ell, s}$. Since Fr_m is semi-simple, for $\ell \neq p$, the image of $I_\ell \otimes \overline{\mathbb{Q}_\ell}$ in $\mathrm{SO}(\mathbf{V}_{\ell, s}, \overline{\mathbb{Q}_\ell})$ is the product

$$\mathrm{SO}(\mathbf{V}_{\ell, s, \overline{\mathbb{Q}_\ell}}^{\mathrm{Fr}_m=1}) \times \prod_{i=1}^r \mathrm{GL}(\mathbf{V}_{\ell, s, \overline{\mathbb{Q}_\ell}}^{\mathrm{Fr}_m=\alpha_i}).$$

From this description, the lemma is immediate. □

7.9. Let $\underline{\mathrm{Aut}}^\circ(A_s^{\mathrm{KS}})$ be the group scheme of units in the ring $\mathrm{End}(A_s^{\mathrm{KS}})_{(p)} \otimes \mathbb{Q}$: this is an algebraic group over \mathbb{Q} . For $\ell \neq p$, there is a natural embedding of algebraic \mathbb{Q}_ℓ -groups

$$i_\ell : \underline{\mathrm{Aut}}^\circ(A_s^{\mathrm{KS}}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow \mathrm{GL}(\mathbf{H}_{\ell, s})$$

defined by the functoriality of ℓ -adic homology.

Similarly, we have a natural embedding of algebraic $\mathbb{Q}_p^{\mathrm{nr}}$ -groups

$$i_p : \underline{\mathrm{Aut}}^\circ(A_s^{\mathrm{KS}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p^{\mathrm{nr}} \hookrightarrow \mathrm{GL}(\mathbf{H}_{\mathrm{cris}, s, \mathbb{Q}_p^{\mathrm{nr}}}).$$

Let $I \subset \underline{\mathrm{Aut}}^\circ(A_s^{\mathrm{KS}})$ be the largest closed sub-group that maps into $\mathrm{GSpin}(\mathbf{V}_{\ell, s})$ under i_ℓ for each $\ell \neq p$, and into $\mathrm{GSpin}(\mathbf{V}_{\mathrm{cris}, s})$ under i_p .

We will need the following proposition, due to Kisin [Kis].

Proposition 7.10 (Kisin). *Suppose that L is perfect, and that $\ell \neq p$ is a prime such that I_ℓ is split. Then the natural map $I_{\mathbb{Q}_\ell} \rightarrow I_\ell$ is an isomorphism.*

□

Remark 7.11. Let us briefly recall the striking yet simple idea behind the proof of the proposition. I is easily seen to be reductive, since it preserves a polarization on A_s^{KS} . So it suffices to show that $I_{\mathbb{Q}_\ell}$ contains a Borel sub-group of I_ℓ . For this, using the splitness of I_ℓ , it is enough to prove that the ℓ -adic space $I(\mathbb{Q}_\ell) \backslash I_\ell(\mathbb{Q}_\ell)$ is compact. Kisin accomplishes this by showing that, for an appropriate compact open $U_\ell \subset I_\ell(\mathbb{Q}_\ell)$, the double coset space $I(\mathbb{Q}_\ell) \backslash I_\ell(\mathbb{Q}_\ell) / U_\ell$ can be identified with a sub-set of $\mathcal{S}_K(\mathbb{F}_q)$: namely, the ‘ ℓ -power isogeny class’ of s .

Proof of (7.4). By (7.6), we can assume that L is perfect and that $L(A_s^{\text{KS}}) \neq 0$. For $\ell \neq p$, the map

$$L(A_s^{\text{KS}}) \otimes \mathbb{Q}_\ell \rightarrow \mathbf{V}_{\ell,s}^{\text{Fr}_m=1}$$

is a map of $I \otimes \mathbb{Q}_\ell$ -representations, and so, by (7.10), for a particular choice of ℓ , it is in fact a map of I_ℓ -representations. But now, by (7.8), $\mathbf{V}_{\ell,s}^{\text{Fr}_m=1}$ is an irreducible representation of I_ℓ , which implies that this map must be an isomorphism for this choice of ℓ . By (7.5), this finishes the proof of the theorem. \square

The following corollary is inspired from [Fal84, §3].

Corollary 7.12. *Suppose that the ℓ -independence assumption (7.2) holds at every point in $\mathcal{S}_K(\overline{\mathbb{F}}_p)$. Let $s \rightarrow \mathcal{S}_K$ be a point defined over a finitely generated extension of \mathbb{F}_p , and let $\bar{s} \rightarrow \mathcal{S}_K$ be a geometric point above s . Then, for each prime $\ell \neq p$, the natural map*

$$L(A_s^{\text{KS}}) \otimes \mathbb{Q}_\ell \rightarrow \mathbf{V}_{\ell,\bar{s}}^{\text{Aut}(k(\bar{s})/k(s))}$$

is an isometry of \mathbb{Q}_ℓ -quadratic spaces.

Proof. We can assume that $k(s) = k(X)$ is the function field of a smooth, geometrically connected variety X over \mathbb{F}_q equipped with an \mathbb{F}_q -valued rational point x_0 . We can also arrange things so that s arises from a map $\tilde{s} : X \rightarrow \mathcal{S}_K$, and thus specializes to an \mathbb{F}_q -valued point $s_0 = \tilde{s} \circ x_0$. By shrinking X if necessary, we can further assume that

$$\text{End}(A_s^{\text{KS}})_{(p)} = \text{End}(A_{s_0}^{\text{KS}})_{(p)}.$$

By the definition of specialness, we have:

$$(7.12.1) \quad L(A_s^{\text{KS}}) = \text{End}(A_s^{\text{KS}})_{(p)} \cap L(A_{s_0}^{\text{KS}}) \subset \text{End}(A_{s_0}^{\text{KS}})_{(p)}.$$

Therefore from (7.4), we find, for a geometric point \bar{s}_0 lying above s_0 :

$$(7.12.2) \quad L(A_s^{\text{KS}}) \otimes \mathbb{Q}_\ell = \left(\text{End}(A_s^{\text{KS}})_{(p)} \otimes \mathbb{Q}_\ell \right) \cap \mathbf{V}_{\ell,\bar{s}_0} \subset \text{End}(\mathbf{H}_{\ell,\bar{s}_0}).$$

By [Zar76], we have, for any $\ell \neq p$:

$$(7.12.3) \quad \text{End}(A_s^{\text{KS}})_{(p)} \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \text{End}_{\text{Aut}(k(\bar{s})/k(s))}(\mathbf{H}_{\ell,\bar{s}}).$$

Combining this with (7.12.2) gives us the result. \square

Finally, we can prove an ℓ -independence result for special endomorphisms.

Corollary 7.13. *Suppose again that the ℓ -independence assumption (7.2) holds at every point in $\mathcal{S}_K(\overline{\mathbb{F}}_p)$. Let T be an \mathcal{S}_K -scheme and suppose that $f \in \text{End}(A_T^{\text{KS}})_{(p)}$. Then f is special if and only if it is ℓ_0 -special for some prime ℓ_0 .*

Proof. Suppose that, for some prime ℓ_0 , f is ℓ_0 -special. Then, by (4.4) and (7.4), the locus where f is special is an open and closed sub-scheme of T that contains $T \otimes \mathbb{Q}$ and $T(\overline{\mathbb{F}}_p)$, and so must be all of T . \square

Remark 7.14. All the above results continue to hold for an arbitrary quadratic lattice (L, Q) satisfying the condition $n \geq r$, provided one restricts to points factoring through a healthy regular scheme T as in (6.28).

8. COMPACTIFICATIONS

We will now show how the results of [MP13a] provide us with toroidal compactifications of the integral models of Spin Shimura varieties. Liberal use will be made of both notation and definitions from [MP13a, §5], but we will give precise references as necessary. A good reference for our particular situation is [Hör10, §10.2]. For references to other works on compactifications, we direct the reader to [MP13a].

(L, Q) will be a quadratic space over $\mathbb{Z}_{(p)}$ such that $V = L_{\mathbb{Q}}$ is non-degenerate of signature $(n, 2)$. We continue to assume $n \geq 1$. We set $r = \dim \text{rad}(L_{\mathbb{F}_p})$. Let $C = C(V, Q)$ be the Clifford algebra over \mathbb{Q} ; when we view it as a representation of $\text{GSpin}(V, Q)$, we will denote it by H . In contrast to prior convention, we will use the letter G to denote the reductive \mathbb{Q} -group $\text{GSpin}(V, Q)$, and we will write $G_{\mathbb{Z}_{(p)}}$ for the smooth $\mathbb{Z}_{(p)}$ -group scheme attached to L by (2.2).

8.1. To begin, we consider the proper *admissible* parabolic sub-groups of G (cf [MP13a, 4.2.2]). For our purposes, justified by [Hör10, §10.2], these are simply the parabolic sub-groups of G that stabilize non-zero isotropic subspaces $M \subset V$. Since $V_{\mathbb{R}}$ has signature $(n, 2)$, the dimension of M can be either 1 or 2.

Fix an isotropic sub-space $M \subset V$, and let $P(M) \subset G$ be the attached parabolic sub-group. We allow $M = 0$, in which case $P(M) = G$. Let $W(M) \subset P(M)$ be its unipotent radical and let $U(M) \subset W(M)$ be its center. In [Pin90, 4.7], Pink associates with $P(M)$ and the Shimura datum (G, X) , a normal sub-group $Q(M) \subset P(M)$. Given a connected component $X^+ \subset X^5$, he also defines a $Q(M)(\mathbb{R})U(M)(\mathbb{C})$ -conjugacy class $F_{M, X^+}^{(2)} \subset \pi_0(X) \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, Q(M)_{\mathbb{C}})$ as follows: There is a natural map

$$\varpi : X \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, Q(M)_{\mathbb{C}})$$

such that the image of X^+ lies within a $Q(M)(\mathbb{R})U(M)(\mathbb{C})$ -orbit. We now take $F_{M, X^+}^{(2)}$ to be the $Q(M)(\mathbb{R})U(M)(\mathbb{C})$ -orbit of $([\mathbf{h}], \varpi_{\mathbf{h}}) \in \pi_0(X) \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, Q(M)_{\mathbb{C}})$, for any $\mathbf{h} \in X^+$.

Let $U(M)(\mathbb{R})(-1)$ be the image of $\text{Lie } U(M)_{\mathbb{R}} \cdot 2\pi\sqrt{-1}$ in $U(M)(\mathbb{C})$. Then $F_{M, X^+}^{(2)}$ admits a map to $U(M)(\mathbb{R})(-1)$ defined as follows: To every $([\mathbf{h}], \omega) \in F_{M, X^+}^{(2)}$ it attaches the unique element $w \in U(M)(\mathbb{R})(-1)$ such that $w\omega w^{-1}$ is defined over \mathbb{R} . Then X^+ can be identified with the pre-image in $F_{M, X^+}^{(2)}$ of an open homogeneous convex self-dual cone $\mathbf{H}_{M, X^+} \subset U(M)(\mathbb{R})(-1)$.

Let $L(M)$ be the Levi quotient of $P(M)$ and let $L(M)_h$ be the image of $Q(M)$ in $L(M)$. If $F_{M, X^+}^{(1)} = U(M)(\mathbb{C}) \backslash F_{M, X^+}^{(2)}$ and $F_{P, X^+} = W(M)(\mathbb{C}) \backslash F_{M, X^+}^{(2)}$, then $(L(M)_h, F_{M, X^+})$ is a Shimura datum. The pair $(Q(M), F_{M, X^+}^{(2)})$ is a **rational boundary component** for (G, X) .

8.2. Let us make these objects more concrete. The case $M = 0$ is trivial, so we begin with the case where $\dim M = 1$. Here, $W(M) = U(M)$ is commutative and can be identified with the vector group attached to the \mathbb{Q} -vector space

$$\text{Hom}(M^{\perp}/M, M) = (M^{\perp}/M) \otimes M.$$

Consider the 3-step ascending filtration

$$0 = W_{-3}V \subset W_{-2}V = W_{-1}V = M \subset W_0V = W_1V = M^{\perp} \subset W_2V = V.$$

This is the unique filtration on V that can be split by a co-character $\mathbb{G}_m \rightarrow G_0$ and is such that the associated ascending filtration on $\text{Lie } G$ is stabilized by $P(M)$ and satisfies: $W_0 \text{Lie } G = \text{Lie } P(M)$ and $W_{-1} \text{Lie } G = W_{-2} \text{Lie } G = \text{Lie } W(M)$.

It induces a 2-step ascending filtration on H (cf. 1.13):

$$0 = W_{-1}H \subset W_0H = \text{im}(M) = W_1H \subset W_2H = H.$$

⁵Note that X has two connected components, which can be thought of as parameterizing the two orientations for \mathbb{C} as an \mathbb{R} -vector space.

Now $Q(M)$ can be identified with the sub-group of $P(M)$ that acts trivially on W_0H . It can be non-canonically identified with the semi-direct product $W(M) \ltimes \mathbb{G}_m$, where \mathbb{G}_m acts on $W(M)$ via dilation. To see this, we choose a basis element $v \in M$ and another isotropic element $v' \in V$ such that $[v, v']_Q = 1$. Then the isomorphism $W(M) \ltimes \mathbb{G}_m \xrightarrow{\cong} Q(M)$ is induced by:

$$\begin{aligned} \mathbb{G}_m &\rightarrow Q(M) \\ z &\mapsto z^2 v' v + v v'. \end{aligned}$$

In fact this homomorphism is precisely the co-character of G that acts trivially on W_0H and via z^2 on $\ker(v')$. In particular, $L(M)_h = \mathbb{G}_m$.

Given $\mathbf{h} \in X$, we have the attached \mathbb{Q} -Hodge structure $V_{\mathbf{h}}$ of weight 0. This gives us a splitting of the weight filtration

$$V_{\mathbb{C}} = M_{\mathbb{C}} \oplus (M_{\mathbb{C}} + V_{\mathbf{h}}^{1,-1})^{\perp} \oplus V_{\mathbf{h}}^{1,-1}.$$

Then $\varpi_{\mathbf{h}} : \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \rightarrow Q_{\mathbb{C}}$ is the unique map such that $\varpi_{\mathbf{h}}(z, w)$ acts via $z^{-1}w^{-1}$ on $M_{\mathbb{C}}$, via zw on $V^{1,-1}$ and trivially on $(M_{\mathbb{C}} + V_{\mathbf{h}}^{1,-1})^{\perp}$.

One can now check that the attached Shimura datum is simply $(\mathbb{G}_m, S^{\pm}(0))$, where $S^{\pm}(0)$ is the set of square roots of -1 in \mathbb{C} [MP13a, 4.1.6]. Note that we are using Pink's slightly more general definition of a (pure) Shimura datum here.

If we choose a basis for M and fix a square root $\sqrt{-1} \in \mathbb{C}$ for -1 , then we can identify $U(M)(\mathbb{R})(-1)$ with $(M^{\perp}/M)_{\mathbb{R}}$, a signature $(n-1, 1)$ quadratic space over \mathbb{R} . Then, given a connected component $X^+ \subset X$, \mathbf{H}_{M,X^+} is a connected component of the spherical cone $\{v \in (M^{\perp}/M)_{\mathbb{R}} : Q(v) < 0\}$.

Lemma 8.3. *Suppose that $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Then the image of $Q(M)(\mathbb{Q}_p) \cap K_p$ in $L(M)_h(\mathbb{Q}_p) = \mathbb{Q}_p^{\times}$ is \mathbb{Z}_p^{\times} .*

Proof. Choose a perfect quadratic lattice (\tilde{L}, \tilde{Q}) admitting (L, Q) as a direct summand, and let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be the attached smooth group scheme over $\mathbb{Z}_{(p)}$. We can identify $G_{\mathbb{Z}_{(p)}}$ as a sub-group scheme of $\tilde{G}_{\mathbb{Z}_{(p)}}$ by (2.2). Set $\tilde{K}_p = \tilde{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$; then, by *loc. cit.* $\tilde{K}_p \cap G(\mathbb{Q}_p) = K_p$. So, to prove the lemma, we can assume that (L, Q) is perfect, and now the assertion is evident. \square

8.4. Next, suppose that $\dim M = 2$. In this case, we have $\mathrm{Lie} U(M) = \wedge^2 M \subset \mathrm{Lie} G$; we will identify $U(M)$ with the vector group attached to $\wedge^2 M$. Moreover, $W(M)/U(M)$ can be identified with the vector group attached to $\mathrm{Hom}(M^{\perp}/M, M)$.

Just as in the previous case, we have ascending filtrations attached to M :

$$\begin{aligned} 0 &= W_{-2}V \subset W_{-1}V = M \subset W_0V = M^{\perp} \subset W_1V = V; \\ 0 &= W_{-1}H \subset W_0H = \mathrm{im}(\wedge^2 M) \subset W_1H = \mathrm{im}(M) \subset W_2H = H. \end{aligned}$$

Then $Q(M) \subset P(M)$ is again the sub-group acting trivially on W_0H . To be more explicit, fix another isotropic plane $M' \subset V$ that pairs non-degenerately with M . This gives us a splitting of $W_{\bullet}V$:

$$V = V_{-1} \oplus V_0 \oplus V_1,$$

where $V_{-1} = M$, $V_1 = M'$ and $V_0 = (M + M')^{\perp}$. This in turn gives us a splitting of $W_{\bullet}H$: $H = H_0 \oplus H_1 \oplus H_2$. We have an identification

$$\begin{aligned} M \otimes C((M')^{\perp}) &\xrightarrow{\cong} H_1 \\ v \otimes y &\mapsto vy \end{aligned}$$

This allows us to view H_1 as a representation of $\mathrm{GL}(M)$ via its natural action on M and the trivial action on $C((M')^{\perp})$. Consider the map $\mathrm{GL}(M) \rightarrow \mathrm{GL}(H)$ via which $\mathrm{GL}(M)$ acts

trivially on H_0 , via the representation just defined on H_1 , and via the determinant on H_2 . This in fact maps $\mathrm{GL}(M)$ into $Q(M)$ and gives us an isomorphism

$$W(M) \ltimes \mathrm{GL}(M) \xrightarrow{\sim} Q(M).$$

So we find that $L(M)_h = \mathrm{GL}(M)$.

Given $\mathbf{h} \in X$, we obtain a decomposition

$$V_{\mathbb{C}} = (M_{\mathbb{C}} \cap F^0 V_{\mathbf{h}, \mathbb{C}}) \oplus (M_{\mathbb{C}} \cap \overline{F}^0 V_{\mathbf{h}, \mathbb{C}}) \oplus (M_{\mathbb{C}}^{\perp} \cap V_{\mathbf{h}}^{0,0}) \oplus \overline{F}^1 V_{\mathbf{h}, \mathbb{C}} \oplus F^1 V_{\mathbf{h}, \mathbb{C}}.$$

Under this decomposition we have

$$\varpi_{\mathbf{h}}(z, w) = (z^{-1}, w^{-1}, 1, w, z).$$

From this description, we find that, for any connected component $X^+ \subset X$, the Shimura datum $(L(M)_h, F_{M, X^+}) = (\mathrm{GL}(M), S^{\pm}(M))$ is just the classical one giving rise to modular curves. Moreover, X^+ determines an orientation on the line $U(M)_{\mathbb{R}}(-1) \simeq \wedge^2 M_{\mathbb{R}}(-1)$, and $\mathbf{H}_{M, X^+} \subset U(M)_{\mathbb{R}}(-1)$ is simply the positive cone under this orientation.

The proof of the following lemma is completely analogous to that of (8.3).

Lemma 8.5. *Suppose that $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Then the image of $Q(M)(\mathbb{Q}_p) \cap K_p$ in $L(M)_h(\mathbb{Q}_p) = \mathrm{GL}(M)(\mathbb{Q}_p)$ is $\mathrm{GL}(M \cap L)(\mathbb{Z}_p)$.*

□

8.6. A cusp label representative or CLR Φ for (G, X) is a triple $(M_{\Phi}, X_{\Phi}^+, g_{\Phi})$, where $M_{\Phi} \subset V$ is an isotropic sub-space, $X_{\Phi}^+ \subset X$ is a connected component and $g_{\Phi} \in G(\mathbb{A}_f)$. Given a compact open $K \subset G(\mathbb{A}_f)$, two CLR's Φ_1, Φ_2 are **equivalent at level K** if there exists $\gamma \in G(\mathbb{Q})$ such that $\gamma \cdot M_{\Phi_1} = M_{\Phi_2}$, $\gamma \cdot X_{\Phi_1}^+ = X_{\Phi_2}^+$ and $\gamma g_{\Phi_1} \in Q(M_{\Phi_1})(\mathbb{A}_f) g_{\Phi_2} K$. When Φ_1, Φ_2, γ satisfy these conditions, we will write $\Phi_1 \xrightarrow{\gamma} \Phi_2$. We will denote by **Cusp $_K(G, X)$** the set of equivalence classes of CLR's at level K . Note that there is exactly one equivalence class $[\Phi]$ with $M_{\Phi} = 0$; we call this the **improper class**.

Fix a compact open K as above; we will assume that K is neat. Given a CLR Φ , we can attach to it the Shimura variety (defined over \mathbb{Q}):

$$\mathrm{Sh}_{K_{\Phi}} := \mathrm{Sh}_{K_{\Phi}}(L(M_{\Phi})_h, F_{\Phi}).$$

Here, $F_{\Phi} := F_{M_{\Phi}, X_{\Phi}^+}$ and K_{Φ} is the image in $L(M_{\Phi})_h$ of $K_{\Phi}^{(2)} = Q(M_{\Phi})(\mathbb{A}_f) \cap g_{\Phi} K g_{\Phi}^{-1}$.

Let $\mathbf{B}_{\Phi} = U(M_{\Phi})(\mathbb{Q}) \cap K_{\Phi}^{(2)}$: this is a free abelian group. Let \mathbf{E}_{Φ} be the \mathbb{Z} -torus with co-character group \mathbf{B}_{Φ} . Over $\mathrm{Sh}_{K_{\Phi}}$, we have an abelian scheme C_{Φ} ⁶, and over C_{Φ} there exists an \mathbf{E}_{Φ} -torsor ξ_{Φ} . We have identifications of analytic varieties:

$$\begin{aligned} C_{\Phi}(\mathbb{C}) &= Q(M_{\Phi})(\mathbb{Q}) \backslash F_{\Phi}^{(1)} \times Q(M_{\Phi})(\mathbb{A}_f) / K_{\Phi}^{(2)}; \\ \xi_{\Phi}(\mathbb{C}) &= Q(M_{\Phi})(\mathbb{Q}) \backslash F_{\Phi}^{(2)} \times Q(M_{\Phi})(\mathbb{A}_f) / K_{\Phi}^{(2)}. \end{aligned}$$

The tower $\xi_{\Phi} \rightarrow C_{\Phi} \rightarrow \mathrm{Sh}_{K_{\Phi}}$ is determined up to isomorphism by the class of Φ in **Cusp $_K(G, X)$** .

8.7. Choose a perfect quadratic $\mathbb{Z}_{(p)}$ -lattice (\tilde{L}, \tilde{Q}) admitting (L, Q) as a direct summand. Let (\tilde{G}, \tilde{X}) and $\tilde{G}_{\mathbb{Z}_{(p)}}$ be the Shimura datum and the smooth $\mathbb{Z}_{(p)}$ -model for \tilde{G} attached to \tilde{L} , respectively. Set $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ and $\tilde{K}_p = \tilde{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Let $K^p \subset G(\mathbb{A}_f^p)$ and $\tilde{K}^p \subset \tilde{G}(\mathbb{A}_f^p)$ be compact open sub-groups such that, with $K = K_p K^p$ and $\tilde{K} = \tilde{K}_p \tilde{K}^p$, $K \subset \tilde{K}$, and the map $\mathrm{Sh}_K \rightarrow \mathrm{Sh}_{\tilde{K}}$ is a closed embedding. Such compact open sub-groups always exist; cf. [Kis10, 2.1.2]. $\mathrm{Sh}_{\tilde{K}}$ admits a smooth integral canonical model $\widetilde{\mathcal{S}}_{\tilde{K}}$ over $\mathbb{Z}_{(p)}$, and, for any

⁶An honest abelian scheme, and not just one up to prime-to- p isogeny.

compatible rppcd $\tilde{\Sigma}$ for $(\tilde{G}, \tilde{X}, \tilde{K})$, this model admits a good toroidal compactification $\mathcal{S}_{\tilde{K}}^{\tilde{\Sigma}}$; cf. [MP13a, 4.6.13].

In particular, for any CLR $\tilde{\Phi}$ for $(\tilde{G}, \tilde{X}, \tilde{K})$, the associated tower $\xi_{\tilde{\Phi}} \rightarrow C_{\tilde{\Phi}} \rightarrow \text{Sh}_{\tilde{K}_{\tilde{\Phi}}}$ admits an integral canonical model over $\mathbb{Z}_{(p)}$:

$$(8.7.1) \quad \Xi_{\tilde{\Phi}} \rightarrow \mathbf{C}_{\tilde{\Phi}} \rightarrow \mathcal{S}_{\tilde{K}_{\tilde{\Phi}}}.$$

Here, $\mathcal{S}_{\tilde{K}_{\tilde{\Phi}}}$ is the integral canonical model for $\text{Sh}_{\tilde{K}_{\tilde{\Phi}}}$, $\mathbf{C}_{\tilde{\Phi}}$ is an abelian scheme over $\mathcal{S}_{\tilde{K}_{\tilde{\Phi}}}$, $\Xi_{\tilde{\Phi}}$ is an $\mathbf{E}_{\tilde{\Phi}}$ -torsor over $\mathbf{C}_{\tilde{\Phi}}$.

If Φ is a CLR for (G, X, K) giving rise to $\tilde{\Phi}$ [MP13a, 4.2.11], then the tower

$$(8.7.2) \quad \xi_{\Phi} \rightarrow C_{\Phi} \rightarrow \text{Sh}_{K_{\Phi}}$$

maps finitely into the corresponding tower for $\tilde{\Phi}$, and it makes sense to consider the normalization of (8.7.1) in (8.7.2). We can write this normalization as a tower $\Xi_{\Phi} \rightarrow \mathbf{C}_{\Phi} \rightarrow \mathcal{S}_{K_{\Phi}}$.

Lemma 8.8. *Assume that L is maximal and that $n \geq r$. Suppose that Φ is proper. Then $\mathcal{S}_{K_{\Phi}}$ is smooth over $\mathbb{Z}_{(p)}$, \mathbf{C}_{Φ} is an abelian scheme over $\mathcal{S}_{K_{\Phi}}$ and Ξ_{Φ} is an \mathbf{E}_{Φ} -torsor over \mathbf{C}_{Φ} .*

Proof. It is enough to show that $\mathcal{S}_{K_{\Phi}}$ is smooth over $\mathbb{Z}_{(p)}$. The rest of the lemma will follow from [MP13a, 4.4.7, 4.6.14]. The assumption $n \geq r$ is needed because we need the integral canonical model \mathcal{S}_K to be reduced (cf. 6.22).

Suppose first that $\dim M = 1$. Then the attached Shimura variety $\text{Sh}_{K_{\Phi}}$ is the finite étale \mathbb{Q} -scheme attached to the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -set $\mathbb{A}_f^{\times}/\mathbb{Q}^{>0}K_{\Phi}$, equipped with a Galois action via the Artin reciprocity map. From (8.3), we see that $\text{Sh}_{K_{\Phi}}$ is unramified at p and so its integral canonical model is a finite étale $\mathbb{Z}_{(p)}$ -scheme $\mathcal{S}_{K_{\Phi}}$.

Now suppose that $\dim M = 2$. Then $\text{Sh}_{K_{\Phi}}$ is just the modular curve of level K_{Φ} . (8.5) shows that the level at p is the maximal compact sub-group that stabilizes $L_{\Phi} := M \cap g_{\Phi,p}L_{\mathbb{Z}_p}$, and so $\text{Sh}_{K_{\Phi}}$ has an integral canonical model $\mathcal{S}_{K_{\Phi}}$ over $\mathbb{Z}_{(p)}$ that parameterizes elliptic curves up to prime-to- p isogeny equipped with K_{Φ}^p -level structures. \square

Remark 8.9. With a bit more effort, one can also describe \mathbf{C}_{Φ} and Ξ_{Φ} intrinsically. When $\dim M_{\Phi} = 1$, this is particularly simple: C_{Φ} is the trivial abelian scheme over $\text{Sh}_{K_{\Phi}}$, and is extended over $\mathcal{S}_{K_{\Phi}}$ by the trivial abelian scheme \mathbf{C}_{Φ} . Similarly, ξ_{Φ} is the trivial \mathbf{E}_{Φ} -torsor over $\text{Sh}_{K_{\Phi}}$, and so is extended over $\mathcal{S}_{K_{\Phi}}$ by the trivial \mathbf{E}_{Φ} -torsor $\Xi_{\Phi} = \mathbf{E}_{\Phi} \times_{\text{Spec } \mathbb{Z}} \mathcal{S}_{K_{\Phi}}$. When $\dim M_{\Phi} = 2$, let $E \rightarrow \mathcal{S}_{K_{\Phi}}$ be the universal elliptic curve (up to prime-to- p isogeny). One can then identify \mathbf{C}_{Φ} with $\underline{\text{Hom}}(L_{\Phi}^{\perp}/L_{\Phi}, E)$ in the prime-to- p isogeny category.

8.10. Let $M \subset V$ be an isotropic line; then, for any isotropic plane $M' \subset V$ with $M \subset M'$, there is a natural inclusion $U(M') \subset U(M)$. For any connected component $X^+ \subset X$, this embeds \mathbf{H}_{M',X^+} into the closure of \mathbf{H}_{M,X^+} in $U(M)(\mathbb{R})(-1)$. We set

$$\mathbf{H}_{M,X^+}^* = \mathbf{H}_{M,X^+} \bigcup (\cup_{M' \supset M} \mathbf{H}_{M',X^+}),$$

where the second union is over isotropic planes containing M .

If M is an isotropic plane, then we set $\mathbf{H}_{M,X^+}^* = \mathbf{H}_{M,X^+}$.

Recall the notion of a compatible rational polyhedral partial cone decomposition (rppcd, for short) for the triple (G, X, K) : This is an assignment Σ , to each CLR Φ , of a collection Σ_{Φ} of rational polyhedral cones $\sigma \subset \mathbf{H}_{\Phi}^* = \mathbf{H}_{M,X^+}^*$ such that any face of a cone in Σ_{Φ} is again in Σ_{Φ} . It has to satisfy certain compatibility and finiteness conditions, for which we direct the reader to [MP13a, 4.2.14]. Let $\Sigma_{\Phi}^{\circ} \subset \Sigma_{\Phi}$ be the set of cones contained in \mathbf{H}_{Φ} . If $\dim M_{\Phi} = 2$, then $\Sigma_{\Phi}^{\circ} = \Sigma_{\Phi}$.

Fix an rppcd Σ for (G, X, K) . We will say that two pairs (Φ_1, σ_1) and (Φ_2, σ_2) with $\sigma_i \in \Sigma_{\Phi_i}^{\circ}$ are **equivalent** if there exists Φ_1 and Φ_2 are equivalent along some $\gamma \in G(\mathbb{Q})$ such that $\gamma \cdot \sigma_1 =$

σ_2 . Here, we are using the natural conjugation isomorphism $\gamma : U(M_{\Phi_1}) \xrightarrow{\sim} U(M_{\Phi_2})$. Write $\mathbf{Cusp}_K^\Sigma(G, X)$ for the set of equivalence classes. We say that an equivalence class $[(\Phi_1, \sigma_1)]$ is a **face** of an equivalence class $[(\Phi_2, \sigma_2)]$ if we can find representatives (Φ_1, σ_1) and (Φ_2, σ_2) such that $M_{\Phi_2} \subset M_{\Phi_1}$ and such that σ_1 is a face of σ_2 when viewed as rational polyhedral cones within $\mathbf{H}_{\Phi_2}^*$.

Given a CLR Φ and a rational polyhedral cone $\sigma \subset \mathbf{H}_\Phi$, we can define the twisted torus embedding $\Xi_\Phi \hookrightarrow \Xi_\Phi(\sigma)$ over \mathbf{C}_Φ ; cf. [MP13a, §4.2]. Let $\mathbf{Z}_\Phi(\sigma) \subset \Xi_\Phi(\sigma)$ be the closed stratum and let $\mathfrak{X}_\Phi(\sigma)$ be the completion of $\Xi_\Phi(\sigma)$ along $\mathbf{Z}_\Phi(\sigma)$.

The following theorem is now an easy consequence of [MP13a, 4.4.7] and (6.22).

Theorem 8.11. *Assume that L is maximal and that $n \geq r^7$. Let $\tilde{\Sigma}$ be a compatible rppcd for $(\tilde{G}, \tilde{X}, \tilde{K})$ inducing the compatible rppcd Σ for (G, X, K) [MP13a, 4.2.20]. Let \mathcal{S}_K be the integral canonical model for Sh_K over $\mathbb{Z}_{(p)}$. Let \mathcal{S}_K^Σ be the normalization of the Zariski closure of Sh_K in $\tilde{\mathcal{S}}_K^\Sigma$, and let \mathbf{D}_K^Σ be the reduced closed sub-scheme underlying $\mathcal{S}_K^\Sigma \setminus \mathcal{S}_K$. Then:*

(1) \mathbf{D}_K^Σ is a relative Cartier divisor over $\mathbb{Z}_{(p)}$. In particular, $\mathcal{S}_{K, \mathbb{F}_p}$ is an open dense sub-scheme of $\mathcal{S}_{K, \mathbb{F}_p}^\Sigma$.

(2) \mathcal{S}_K^Σ admits a stratification indexed by $\mathbf{Cusp}_K^\Sigma(G, X)$:

$$\mathcal{S}_K^\Sigma = \bigsqcup_{[(\Phi, \sigma)]} \mathbf{Z}_{[(\Phi, \sigma)]}.$$

In this stratification, $\mathbf{Z}_{[(\Phi, \sigma)]}$ lies in the closure of $\mathbf{Z}_{[(\Phi', \sigma')]}$ if and only if $[(\Phi, \sigma)]$ is a face of $[(\Phi', \sigma')]$.

(3) For any representative (Φ, σ) of $[(\Phi, \sigma)]$, $\mathbf{Z}_{[(\Phi, \sigma)]}$ is isomorphic to $\mathbf{Z}_\Phi(\sigma)$. Moreover, the completion $\mathfrak{X}_{[(\Phi, \sigma)]}$ of \mathcal{S}_K^Σ along $\mathbf{Z}_{[(\Phi, \sigma)]}$ is compatibly isomorphic to $\mathfrak{X}_\Phi(\sigma)$.

□

Remark 8.12. The argument in [MP13a, §4.7] now shows that, when L is maximal, the Hecke action of $G(\mathbb{A}_f^p)$ on the integral canonical model \mathcal{S}_{K_p} extends to the tower of its toroidal compactifications in the expected way.

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⁷This will always be the case, unless $n = 1$ and $r = 2$.

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